Binary relations:

- binary relations are ubiquitous in math
- e.g. we have order relations like:

$$x \leq y$$
$$x < y$$

- the subset relation

$$X \subseteq Y$$

- the divisibility relation

$$n \mid m \qquad (\text{"}n \text{ divides } m\text{"})$$

↳ all assert a relation between two mathematical objects (hence "binary")

↳ but what are $\leq$, $<$, $\subseteq$, etc. as math'l objects themselves?

↳ we will define binary relations as sets of ordered pairs.

Def'n. - Sps A and B are sets. A binary relation on A and B is just a subset $R \subseteq A \times B$.

- if $(a,b) \in R$ we say "a is related to b" and sometimes write aRb.

- for a relation $R \subseteq A \times B$, we say A is the <u>domain</u> of R, B is the <u>codomain</u>.
- frequently, $A = B$, so that $R \subseteq A \times A$. In this case we say: R is a <u>relation</u> <u>on</u> <u>A</u>.

<u>Ex</u>'s: ① - Let A = set of Shakespeare's characters
B = " " " plays.

- Define a relation $R \subseteq A \times B$ by:
$(a,b) \in R$ iff a appears in b

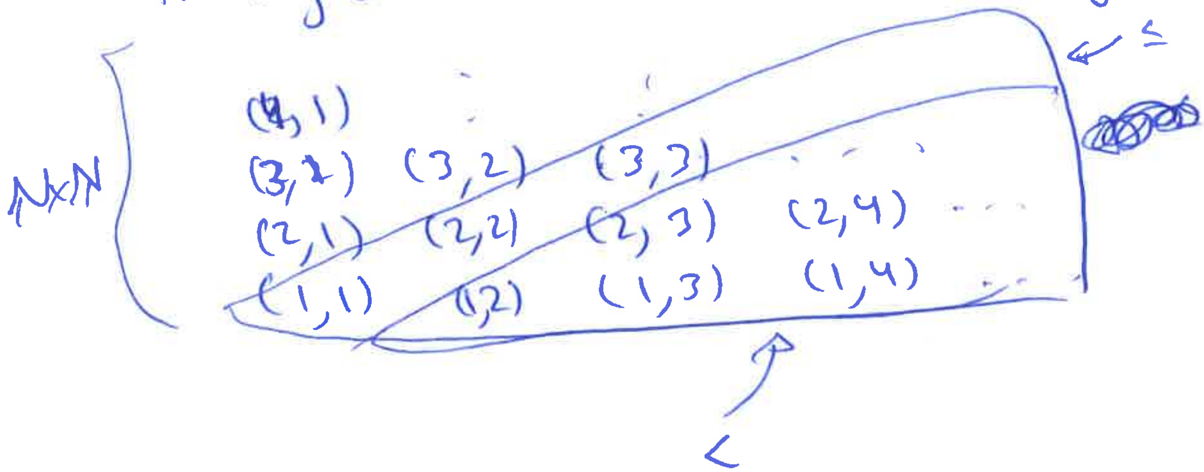
(or: using set-builder:
$R = \{(a,b) \in A \times B \mid a \text{ appears in } b\}$)

- then:
(Romeo, "Romeo and Juliet") $\in R$
(Iago, "Othello") $\in R$
(Romeo, "Othello") $\notin R$
- might also write:
Romeo R "Romeo and Juliet"
Romeo $\not R$ "Othello".

② Consider $\leq$ and $<$ as relations on $\mathbb{N}$. Can think of them as sets of pairs:

$\leq = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ is not greater than } b\}$
$= \{(1,1), (1,2), (5,217), \ldots\}$
$< = \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ is strictly less than } b\}$
$= \{(1,2), (17,200), \ldots\}$

- we write $1 \leq 2$ instead of $(1,2) \in \leq$ but these mean the same thing.
- likewise $2 \not< 1$ means $(2,1) \notin <$.
- can visualize $\leq$ and $<$ as "right-lower-triangular" subsets of the grid $\mathbb{N} \times \mathbb{N}$:



③ Let A be a set. Can think of $=$ as a binary relation on A:

$=$ is the set $\{(x,y) \in A \times A \mid x = y\}$,

i.e. $\{(x,x) : x \in A\}$.

④ relations are <u>arbitrary</u> sets of pairs, and need not be defined by some intelligible property.

e.g. — $R = \{(1,1), (2,\pi), (3,\sqrt{2})\} \subseteq \mathbb{R} \times \mathbb{R}$

is a relation on $\mathbb{R}$.

— $\emptyset$ is a relation on <u>every</u> set A ("the empty relation")

Properties relations can have:

Def'n Spr A is a set and $R \subseteq A \times A$ is a relation on A.

1. R is reflexive iff $(\forall x \in A)((x,x) \in R)$
2. R is symmetric iff $(\forall x,y \in A)\ ((x,y) \in R \Rightarrow (y,x) \in R)$
3. R is transitive iff $(\forall x,y,z \in A)\ ((x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R)$
4. R is antisymmetric iff $(\forall x,y \in A)\left((x,y) \in R \wedge (y,x) \in R \Rightarrow x = y\right)$

Ex's ① on any set A, the equality relation $=$ is reflexive, symmetric, and transitive

(... and also antisymmetric, trivially)

↳ relations w/ these three properties are called equivalence relations (more later...)

② $\leq$ (e.g. on $\mathbb{N}$) is reflexive, transitive, and antisymmetric.

why: $(\forall n \in \mathbb{N})\ (n \leq n)$ ✓

$(\forall n,m,\ell \in \mathbb{N})\ (n \leq m \text{ and } m \leq \ell \Rightarrow n \leq \ell)$ ✓

$(\forall n,m \in \mathbb{N})\ (n \leq m \text{ and } m \leq n \Rightarrow m = n)$ ✓

but $\leq$ is not symmetric, since e.g.

$3 \leq 5$ but $5 \not\leq 3$.

(3) $<$ (e.g. on $\mathbb{N}$) is not reflexive or symmetric, but is transitive

(is $<$ antisymmetric?
Yes ... why?)

(4) Let $A = \{\text{rock}, \text{paper}, \text{scissors}\}$
define a relation R on A by $(a,b) \in R$
iff a beats b.

Then R is not transitive since:

$(\text{scissors}, \text{paper}) \in R$
$(\text{paper}, \text{rock}) \in R$
but $(\text{scissors}, \text{rock}) \notin R$.

(5) Consider the divisibility relation $|$ on $\mathbb{N}$: defined by $n | m$ iff n divides m
i.e. $(\exists k \in \mathbb{N})(m = nk)$

e.g. $2 | 4$ and $2 | 12$ but $2 \nmid 7$.

Claim: the divisibility relation $|$ on $\mathbb{N}$ is:

(1) reflexive
(2) not symmetric
(3) transitive
(4) antisymmetric

(6)

Pf ① Fix $n \in \mathbb{N}$ arbitrarily. Then $n|n$ since $n = n \cdot 1$.

② e.g. $2|4$ but $4 \nmid 2$.

③ Fix $n, m, \ell \in \mathbb{N}$ and sps $n|m$ and $m|\ell$.

i.e. $\exists k_1, k_2 \in \mathbb{N}$ s.t. $m = nk_1$

$\ell = m k_2$

but then $\ell = m k_2 = (nk_1)k_2 = n(k_1 k_2)$

so $n|\ell$. ✓

④ Fix $n, m \in \mathbb{N}$ and sps $n|m$ and $m|n$

then $m = k_1 n$

$n = k_2 m$ $\Rightarrow$ $m = k_1 k_2 m$

$\Rightarrow$ $k_1 k_2 = 1$

$\Rightarrow$ $k_1 = k_2 = 1$

$\Rightarrow$ $n = m$.

⑥ Now consider divisibility $|$ on $\mathbb{Z}$, i.e. for $n, m \in \mathbb{Z}$

$n|m$ iff $\exists k \in \mathbb{Z}$ s.t. $m = nk$.

Then: $|$ remains reflexive, transitive.

Still antisymmetric? no: e.g. $2|-2$

$-2|2$

but $2 \neq -2$.

Equivalence relations

Def'n A relation R on a set A is called an <u>equivalence relation</u> iff R is reflexive, symmetric, and transitive.

<u>Ex's</u> ① Given a set A, the equality relation $=$ is an equivalence relation on A.

Pf: $\forall x, y, z \in A$ we have:

(i) $x = x$ ✓

(ii) $x = y \Rightarrow y = x$ ✓

(iii) $x = y \wedge y = z \Rightarrow x = z$ ✓

② <u>Recall</u>: the floor of a real number x, denoted $\lfloor x \rfloor$ is the unique integer n s.t. $n \leq x < n+1$.

e.g. $\lfloor 1.5 \rfloor = 1$

$\lfloor \pi \rfloor = 3$

$\lfloor -2.67 \rfloor = -3$

$\lfloor 5 \rfloor = 5$.

Define a relation R on $\mathbb{R}$ by:

$(x, y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$

i.e. $R = \{(x, y) \in \mathbb{R}^2 \mid \lfloor x \rfloor = \lfloor y \rfloor\}$.

<u>Claim</u>: R is an equivalence relation

<u>Pf</u>: (i) Fix $x \in \mathbb{R}$. Then $\lfloor x \rfloor = \lfloor x \rfloor$. Hence $(x, x) \in R$. Since x was arbitrary we have $(\forall x \in \mathbb{R})\ (x, x) \in R$.

(ii) Fix $x, y \in \mathbb{R}$.

Sps $(x,y) \in R$.

Then $\lfloor x \rfloor = \lfloor y \rfloor$.

But then $\lfloor y \rfloor = \lfloor x \rfloor$ too

So $(y,x) \in R$ ✓

(iii) Fix $x, y, z \in \mathbb{R}$.

Sps $(x,y) \in R$ and $(y,z) \in R$.

then $\lfloor x \rfloor = \lfloor y \rfloor$ and $\lfloor y \rfloor = \lfloor z \rfloor$

But then $\lfloor x \rfloor = \lfloor z \rfloor$ too (transitivity of =)

Hence $(x,z) \in R$ ✓

3. More generally, suppose that $f : \mathbb{R} \to \mathbb{R}$ is a particular function.

Define a relation R_f on $\mathbb{R}$ by:

$(x,y) \in R_f$ iff $f(x) = f(y)$

i.e. $R_f = \{(x,y) \in \mathbb{R}^2 \mid f(x) = f(y)\}$

<u>then</u>: R_f is an equivalence relation

Pf: HW.

4. Define a relation $\equiv_3$ on $\mathbb{Z}$ by: $(n,m) \in \equiv_3$ iff $3 \mid (m-n)$

i.e. $\equiv_3 = \{(n,m) \in \mathbb{Z}^2 \mid 3 \mid (m-n)\}$

$\rightsquigarrow$ we'll write $n \equiv_3 m$ instead of $(n,m) \in \equiv_3$.

e.g. $2 \equiv_3 5$ since $3 \mid (5-2)$

$7 \equiv_3 -2$ since $3 \mid (7-(-2))$

$6 \not\equiv_3 7$ since $3 \nmid (7-6)$

<u>Claim</u> $\equiv_3$ is an equiv. relation on $\mathbb{Z}$.

<u>Pf.</u> (i) Fix $n \in \mathbb{Z}$. Observe that $3 \mid n-n$, i.e. $3 \mid 0$, because $0 = 3 \cdot 0$. Thus $n \equiv_3 n$. ✓

(ii) Fix $n, m \in \mathbb{Z}$ and spse $n \equiv_3 m$. We prove $m \equiv_3 n$.

<u>Pf</u>: Since $n \equiv_3 m$ we have $3 \mid m-n$, i.e. $\exists k \in \mathbb{Z}$ s.t. $m-n = 3k$.

But then $n-m = 3(-k)$

So $3 \mid n-m$

So $m \equiv_3 n$ ✓

(iii) Fix $n, m, l \in \mathbb{Z}$. Sps $n \equiv_3 m$ and $m \equiv_3 l$. We prove $n \equiv_3 l$.

<u>Pf</u>: we know $\exists k_1, k_2 \in \mathbb{Z}$ s.t.

$$m - n = 3k_1$$
$$l - m = 3k_2.$$

adding these equations gives:

$$(m-n) + (l-m) = 3k_1 + 3k_2$$
$$\Rightarrow l - n = 3(k_1+k_2)$$
$$\Rightarrow 3 \mid l-n, \text{ i.e. } n \equiv_3 l. ✓$$

↪ $\equiv_3$ is called equivalence modulo 3.
↪ more common to write

$n \equiv m \pmod 3$ instead of $n \equiv_3 m$.

↪ another way to think about it:
$n \equiv m \pmod 3$ iff n, m have the same remainder when divided by 3.

e.g. $2 \equiv 5 \pmod 3$

since $2 = 3 \cdot 0 + 2$ } same remainder
$5 = 3 \cdot 1 + 2$

$7 \equiv 13 \pmod 3$

since $7 = 3 \cdot 2 + 1$ } same
$13 = 3 \cdot 4 + 1$

$7 \equiv -2 \pmod 3$

since $7 = 3 \cdot 2 + 1$
$-2 = 3 \cdot (-1) + 1$

but $7 \not\equiv 11 \pmod 3$

since $7 = 3 \cdot 2 + 1$ } diff
$11 = 3 \cdot 3 + 2$

(5) Nothing special about 3. For any fixed $k \in \mathbb{N}$, can define $\equiv_k$ on $\mathbb{Z}$ by:

$n \equiv_k m$ iff $k \mid m - n$.

(iff n, m have same remainder when divided by k)

↳ again more common to write
$n \equiv m \pmod{k}$ instead of $n \equiv_k m$.

↳ all of these "congruence modulo k" relations are equivalence relations.

Nonexamples of equiv. relations.

① Consider $\leq$ (e.g. on $\mathbb{R}$): is reflexive, transitive, but not symmetric hence not an equiv. relation.

② Consider the relation $\neq$ of inequality on $\mathbb{Z}$.

is symmetric, since $n \neq m \Rightarrow m \neq n$.
but not reflexive (in fact: never true that $n \neq n$)
nor transitive (e.g. $2 \neq 4$ and $4 \neq 2$, but $2 = 2$).

Equivalence Classes

– Sps R is an equivalence relation on a set A.

Def'n For each $x \in A$, the equivalence class of x, denoted $[x]_R$, is the set of el'ts related to x by R, i.e.

$$[x]_R = \{y \in A \mid (x,y) \in R\}$$

(Note: by symmetry could have defined:

$$[x]_R = \{y \in A \mid (y, x) \in R\}.)$$

Warning: - overloaded notation: we've used $[\ ]$'s when defining $[n] = \{1, \dots, n\}$.
- this is completely unrelated to meaning of $[x]_R$ for an equiv. relation R. - so don't get confused!

Ex's ① Let $=$ denote the equality relation on $\mathbb{N}$. Then for any fixed $n \in \mathbb{N}$ we have:

$$[n]_= \text{ is } \{m \in \mathbb{N} \mid n = m\}$$
$$\text{i.e. } \{n\}.$$

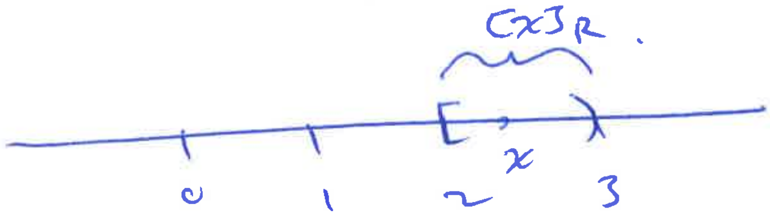
(so $[1]_=$ is $\{1\}$ $[2]_=$ is $\{2\}$ etc.)

② Let R denote the floor equiv. relation on $\mathbb{R}$, i.e. $(x, y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

Now: Fix $x \in \mathbb{R}$ and sps $\lfloor x \rfloor = n$

Then:
$$\begin{aligned} [x]_R &= \{y \in \mathbb{R} \mid (x, y) \in R\} \\ &= \{y \in \mathbb{R} \mid \lfloor x \rfloor = \lfloor y \rfloor\} \\ &= \{y \in \mathbb{R} \mid n = \lfloor y \rfloor\} \\ &= \{y \in \mathbb{R} \mid n \le y < n+1\} \\ &= [n, n+1) \end{aligned}$$

Pic: e.g. if $x = 2.34$ then $\lfloor x \rfloor = 2$
and so $[x]_R = [2,3)$



Notice: the equiv. classes of R form a partition of $\mathbb{R}$

we'll prove later this always happens.

(3) Consider $\equiv_3$, equivalence mod 3 on $\mathbb{Z}$.
Q: what are the equiv. classes?
Let's write some down.

$$[0]_{\equiv_3} \text{ is } \{n \in \mathbb{Z} \mid 0 \equiv_3 n\}$$
$$= \{n \in \mathbb{Z} \mid 3 \mid (n-0)\}$$
$$= \{n \in \mathbb{Z} \mid 3 \mid n\}$$
$$= \{\ldots -3, 0, 3, 6, \ldots\}$$

$$[1]_{\equiv_3} \text{ is } \{n \in \mathbb{Z} \mid 1 \equiv_3 n\}$$
$$= \{n \in \mathbb{Z} \mid 3 \mid (n-1)\}$$
$$= \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\ n = 3k+1\}$$
$$= \{\ldots, -2, 1, 4, 7, \ldots\}$$

$$[2]_{\equiv_3} \text{ is } \{n \in \mathbb{Z} \mid 2 \equiv_3 n\}$$
$$= \{n \in \mathbb{Z} \mid 3 \mid n-2\}$$
$$= \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\, n = 3k+2\}$$
$$= \{\ldots -1, 2, 5, 8, \ldots\}.$$

$$[3]_{\equiv_3} \text{ is } \{n \in \mathbb{Z} \mid 3 \equiv_3 n\}$$
$$= \{n \in \mathbb{Z} \mid 3 \mid n-3\}$$
$$= \{n \in \mathbb{Z} \mid 3 \mid n\}$$
$$= \{\ldots -3, 0, 3, 6, \ldots\}$$
$$= [0]_{\equiv_3}.$$

Sim'ly we can check:

$$[4]_{\equiv_3} = [1]_{\equiv_3}$$
$$[5]_{\equiv_3} = [2]_{\equiv_3}$$
$$[6]_{\equiv_3} = [3]_{\equiv_3} = [0]_{\equiv_3} \quad \text{etc.}$$

<u>Notice</u>: – equiv classes consist of all $n \in \mathbb{Z}$ of a given remainder when divided by 3 – so there is one class for each possible remainder 0, 1, 2.

– again: the equiv. classes form a <u>partition</u> of $\mathbb{Z}$:

pairwise disjoint

$$\mathbb{Z} = \{\ldots -3, 0, 3, 6, \ldots\} \cup \{\ldots, -2, 1, 4, 7, \ldots\} \cup \{\ldots -1, 2, 5, 8, \ldots\}$$

$= [0]_{\equiv_3} \cup [1]_{\equiv_3} \cup [2]_{\equiv_3}$.

<u>Notation</u>: for equivalence modulo k, we'll write $[x]_k$ instead of $[x]_{\equiv_k}$.

e.g. we'll abbreviate above as:

$$\mathbb{Z} = [0]_3 \cup [1]_3 \cup [2]_3.$$

---

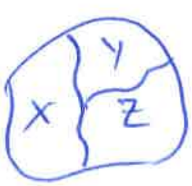
our next goal is to see that "partition" and "equivalence relation" are, in a sense, two names for the same concept.

<u>Recall</u>: if A is a set, a <u>partition</u> $\mathbb{P}$ of A is a collection of subsets of A (i.e. $\mathbb{P} \subseteq P(A)$) s.t.

1. $(\forall X \in \mathbb{P})\ X \neq \emptyset$.
2. $(\forall X, Y \in \mathbb{P})\ (X \neq Y \Rightarrow X \cap Y = \emptyset)$
3. $\bigcup_{X \in \mathbb{P}} X = A$.

note: (2) says the pieces of the partition are <u>pairwise disjoint</u>: can also write this condition as: $(\forall X, Y \in \mathbb{P})(X = Y \vee X \cap Y = \emptyset)$

<u>Picture</u>:



A

$\mathbb{P} = \{X, Y, Z\}$ a partition of A (into 3 pieces) ex's ① Let

$$X = \{\ldots, -3, 0, 3, 6, \ldots\}$$
$$Y = \{\ldots, -2, 1, 4, 7, \ldots\}$$
$$Z = \{\ldots, -1, 2, 5, 8, \ldots\}$$

Then $\mathbb{P} = \{X, Y, Z\}$ is a partition of

Pf: ① $X, Y, Z \neq \emptyset$. ✓

② $X \cap Y = X \cap Z = Y \cap Z = \emptyset$ ✓

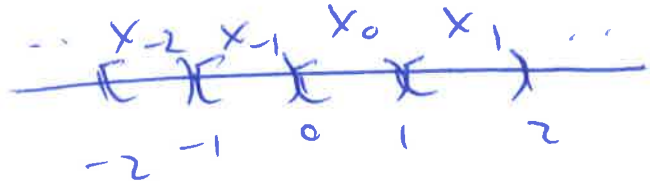
③ $X \cup Y \cup Z = \mathbb{Z}$. ✓

② For every $n \in \mathbb{Z}$, define:

$$X_n = \{y \in \mathbb{R} \mid n \le y < n+1\}$$
$$= [n, n+1)$$

Then $\mathbb{P} = \{X_n : n \in \mathbb{Z}\} = \{\ldots, X_{-1}, X_0, X_1, X_2, \ldots\}$ is a partition of $\mathbb{R}$.

Pf. you try



③ Let $A = \{1, 2, 3, 4\}$

Define $X = \{1\}$ $Y = \{2, 3, 4\}$

Then $\mathbb{P} = \{X, Y\} = \{\{1\}, \{2, 3, 4\}\}$ is a partition of A.

Equivalence classes partition sets.

Def'n: Sps R is an equiv. relation on A. We denote the set of equiv. classes of R as A/R:

i.e. $A/R = \{[x]_R : x \in A\}$

read "A mod R"

Ex's – Consider $\equiv_3$ on $\mathbb{Z}$.

Then: $\mathbb{Z}/\equiv_3 = \{[n]_3 : n \in \mathbb{Z}\}$

$= \{\ldots, [-1]_3, [0]_3, [1]_3, [2]_3, \ldots\}$

We already checked:

$\ldots = [-3]_3 = [0]_3 = [3]_3 = [6]_3 = \ldots$

$\ldots = [-2]_3 = [1]_3 = [4]_3 = \ldots$

$\ldots = [-1]_3 = [2]_3 = [5]_3 = \ldots$

think: "set of possible remainders mod 3"

So really: $\mathbb{Z}/\equiv_3 = \{[0]_3, [1]_3, [2]_3\}$

could as well write:

$\mathbb{Z}/\equiv_3 = \{[3]_3, [-2]_3, [5]_3\}$.

Notation: - it is conventional to write

$\mathbb{Z}/\equiv_n$ as $\mathbb{Z}/n\mathbb{Z}$.

- just like w/ 3, in general we have:

$$\mathbb{Z}/\equiv_n = \mathbb{Z}/n\mathbb{Z} = \{[0]_n, [1]_n, \ldots, [n-1]_n\}.$$

② Let R be the floor equiv. relation on $\mathbb{R}$: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

- we checked before: equiv. classes are sets of the form $[n, n+1)$

- indeed for any $x \in [n, n+1)$ we have $[x]_R = [n, n+1)$

- so e.g. $[0]_R = [1/2]_R = [.9]_R$

$$= [0, 1)$$

$$[1]_R = [1.2799\ldots]_R = [1.99]_R$$

$$= [1, 2)$$

etc.

- Thus $\mathbb{R}/R = \{[x]_R : x \in \mathbb{R}\}$

$$= \{\ldots, [-1,0), [0,1), [1,2), \ldots\}$$

$$= \{\ldots, [-1]_R, [0]_R, [1]_R, \ldots\}$$

etc. $= \{\ldots [-1/2]_R, [1/2]_R, [1.9]_R, \ldots\}$

Notice: in both examples 1 and 2 the set of equiv. classes forms a partition of the underlying set ($\mathbb{Z}$ in 1, $\mathbb{R}$ in 2)

- turns out this is always the case:

Theorem: If R is an equiv. relation on A, then A/R is a partition of A.

Pf: HW. For a hint, see 6.7.13 on pg. 449, which outlines an approach to the proof.