Induction:

Question: what happens if we add the first several odd positive integers together?

$$1 = 1 = 1^2$$

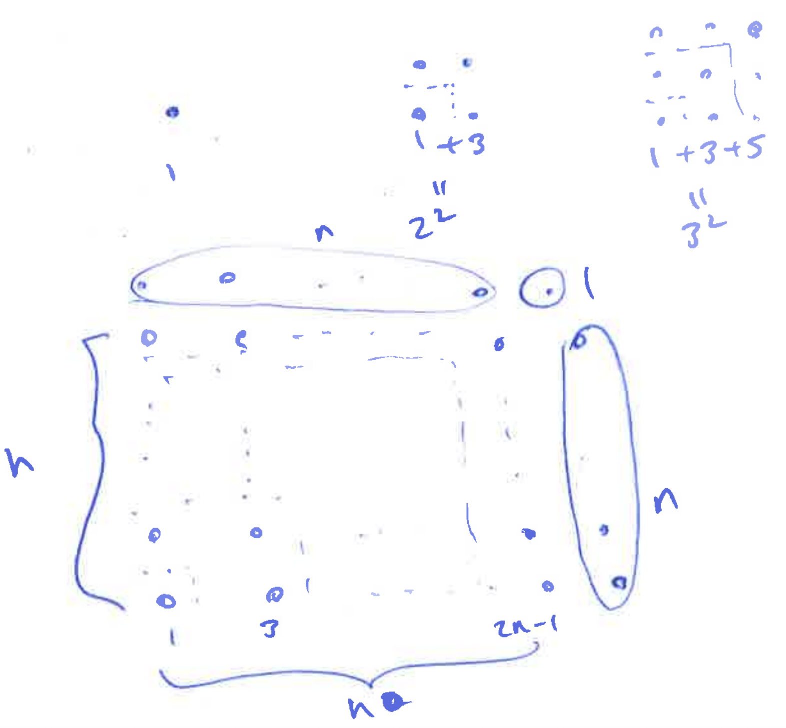
$$1+3 = 4 = 2^2$$

$$1+3+5 = 9 = 3^2$$

$$1+3+5+7 = 16 = 4^2$$

$$(*) \quad 1+3+5+7+\cdots+(2n-1) \overset{??}{=} n^2$$

Picture:



$$n^2 + (2n+1) = (n+1)^2$$

- picture suggests (*) is true for all $n \in \mathbb{N}$. How would we prove it?
- picture also suggests that proof for $n+1$ depends on proof for n.

Theorem: For every $n \in \mathbb{N}$ we have:

$$1+3+5+\dots+(2n-1) = n^2$$

i.e. $$\sum_{k=1}^{n} 2k-1 = n^2$$

PF: - Clearly this is true when $n=1$ since

$$\sum_{k=1}^{1} 2k-1 = 1 = 1^2$$

- So sps $n \in \mathbb{N}$ is fixed and we have that the identity holds for n, i.e. assume that

$$\sum_{1}^{n} 2k-1 = n^2$$

- now consider the sum for $n+1$:

$$\sum_{k=1}^{n+1} 2k-1 = \underbrace{1+3+\cdots+2n-1}\ + 2(n+1)-1$$

$$= \sum_{k=1}^{n} 2k-1 \qquad + 2n+1$$

$$= n^2+2n+1$$

by our assumption

$$= (n+1)^2$$

We've shown:

(a) identity holds for $n=1$

(b) if it holds for a fixed n,
then it holds for $n+1$ too.

But then: since identity holds for $n=1$,
it holds for $n=2$,

⋮

and so for $n=3$,
and so for $n=4$,

⋮

and so for all $n \in \mathbb{N}$!

Validity of this kind of argument is called principle of mathematical induction (PMI) e.g. "$\sum_{1}^{n} 2k-1 = n^2$"

"Theorem" (PMI) Suppose $P(n)$ is a variable proposition. Suppose further that:

① $P(1)$ holds
② $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ holds

then $(\forall n \in \mathbb{N})\, P(n)$ holds.

↳ for a "proof" see the book.
↳ we'll take PMI as an axiom (i.e. we'll assume the type of reasoning we used above is valid)
↳ later, we'll show PMI is equiv. to another intuitively obvious principle.

Using PMI to prove $(\forall n \in \mathbb{N})\, P(n)$

① (Base case) Verify $P(1)$ directly
② (Inductive hypothesis). Fix $n \in \mathbb{N}$ and assume $P(n)$.
③ (Inductive step) using this hypothesis, deduce $P(n+1)$.

PMI says: if you can do ①, ②, ③ then $(\forall n \in \mathbb{N})\, P(n)$ holds.

Ex: 1 What happens if we sum the first n natural numbers?

$$1+2+\cdots+n \overset{??}{=}$$

First few:

$$1 = 1 = \frac{1\cdot 2}{2}$$

$$1+2 = 3 = \frac{2\cdot 3}{2}$$

$$1+2+3 = 6 = \frac{3\cdot 4}{2}$$

$$1+2+3+4 = 10 = \frac{4\cdot 5}{2}$$

$$1+2+\cdots+n \overset{??}{=} \frac{n(n+1)}{2}$$

Theorem: For every $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Pf: Let $P(n)$ be the prop'n

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(we want to prove $(\forall n \in \mathbb{N})\, P(n)$)

Base case: $P(1)$ is true since

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2} \checkmark$$

Inductive hypothesis: Fix $n \in \mathbb{N}$, and assume $P(n)$, i.e. assume

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Inductive step: Now consider

$$\sum_{k=1}^{n+1} k = \underbrace{1+2+\cdots+n}+(n+1)$$

$$= \sum_{k=1}^{n} k + (n+1)$$

by the IH

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$$= \frac{(n+1)[(n+1)+1]}{2}$$

hence $P(n+1)$ holds.

by PMI, $P(n)$ holds for every $n \in \mathbb{N}$

<u>Notice</u>: proof doesn't give insight into how we might have guessed formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

but once we <u>can</u> guess formula, PMI gives a way of verifying that it's really true $\forall n \in \mathbb{N}$.

② (Geometric Series)

Fix $x \in \mathbb{R}$ with $x \neq 0, 1$.
Then for every $n \in \mathbb{N}$ we have:

$$1 + x + x^2 + \dots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

i.e. $$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}$$ ← call this $P(n)$

(BC) $P(1)$ holds since

$$\sum_{k=0}^{0} x^k = x^0 = 1 = \frac{x^1 - 1}{x - 1}$$

since $x \neq 0$ since $x \neq 1$.

(8)

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$ holds, i.e.

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1}$$

(IS) Now consider:

$$\sum_{k=0}^{n} x^k = \sum_{k=0}^{n-1} x^k + x^n$$

$$\overset{\text{by IH}}{=} \frac{x^n - 1}{x-1} + x^n$$

$$= \frac{x^n - 1}{x-1} + \frac{x^n(x-1)}{x-1}$$

$$= \frac{\cancel{x^n} - 1 + x^{n+1} - \cancel{x^n}}{x-1}$$

$$= \frac{x^{n+1} - 1}{x-1}$$

So $P(n+1)$ holds.

by PMI $P(n)$ holds $\forall n \in \mathbb{N}$,

i.e. $\displaystyle\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1} \quad \forall n \in \mathbb{N}$.

(3) Prop'n: For every $n \in \mathbb{N}$, $7^n - 4^n$ is a multiple of 3.

Pf (BC) if $n=1$, statement holds since $7^1 - 4^1 = 3$ ✓

(IH) Fix $n \in \mathbb{N}$ and assume $\exists k \in \mathbb{N}$ s.t. $7^n - 4^n = 3k$

(IS) Now, observe:

$$7^n = 3k + 4^n$$
$$\Rightarrow 7^{n+1} = (3k+4^n)7$$
$$= 21k + 7 \cdot 4^n$$
$$= 21k + (3+4)4^n$$
$$= 21k + 3 \cdot 4^n + 4^{n+1}$$
$$\Rightarrow 7^{n+1} - 4^{n+1} = 21k + 3 \cdot 4^n$$
$$= 3(7k + 4^n)$$
$$= 3M \quad \text{where } M = 7k + 4^n$$

hence $7^{n+1} - 4^{n+1}$ is a multiple of 3. ✓

By PMI, $7^n - 4^n$ is a multiple of 3 for every $n \in \mathbb{N}$. ✓

Variants of induction:

↳ nothing special about $n=1$ as our base case.

Thm (PMI w/ a different base case).

- Sps $P(n)$ is a var. prop'n and $n_0 \in \mathbb{Z}$ is fixed (possibly 0 or negative!)
- Let $S = \{n \in \mathbb{Z} \mid n \geq n_0\} = \{n_0, n_0+1, n_0+2, \dots\}$

If we have

① $P(n_0)$ holds
② $(\forall n \in S)(P(n) \Rightarrow P(n+1))$ holds

then we have

$(\forall n \in S)\, P(n)$ holds.

↳ can prove theorem w/ regular PMI (see book if interested)

↳ proof template nearly the same as with regular PMI:

① (BC) Verify $P(n_0)$
② (IH) Fix $n \in S$ (i.e. fix $n \in \mathbb{Z}$ with $n \geq n_0$) and assume $P(n)$
③ (IS) Prove $P(n+1)$.

Ex: Question. For which $n \in \mathbb{N}$ do we have $n! > 2^n$?

Let's see...

n	$n!$	2^n
1	1	2
2	2	4
3	6	8
4	24	16
5	120	32

Seems like: if $n \geq 4$ then $n! > 2^n$.

Let's prove this.

Prop'n For every $n \in \mathbb{N}$ with $n \geq 4$ we have $n! > 2^n$.

(Note: here, our S would be $\{n \in \mathbb{N} \mid n \geq 4\} = \{4, 5, 6, \ldots\}$

our n_0 is 4).

Pf: Let $P(n)$ be the prop'n "$n! > 2^n$"

(BC) $P(4)$ holds since $4! = 24 > 16 = 2^4$.

(IH) Fix $n \in \mathbb{N}$, $n \geq 4$ [Note: we fix $n \geq 4$, not $n > 4$] and assume $P(n)$ holds, i.e. assume $n! > 2^n$

(IS) Then we have:

$$(n+1)! = n!(n+1)$$
$$> 2^n (n+1) \quad \text{(by IH)}$$
$$> 2^n \cdot 2 \quad \text{(since } n \geq 4 \text{ hence } n+1 \geq 5\text{)}$$
$$= 2^{n+1}$$

So $P(n+1)$ holds.

by induction, we've proved: for every $n \geq 4$ we have $n! > 2^n$.

<u>Induction</u> <u>w/ Jumps</u>:

- sometimes we want to prove $P(n)$, not for <u>all</u> n,
  but when is even,
  or ... when n is odd,
  or ... when n is a multiple of 7, etc.

↳ can still argue inductively.

<u>Thm</u> Let $P(n)$ be a var prop'n.
Fix $n_0 \in \mathbb{Z}$ and $k \in \mathbb{N}$ (n_0 = "starting point", k = "jump")
Let $S = \{n_0, n_0 + k, n_0 + 2k, \ldots\}$

<u>If</u> we have:

① $P(n_0)$
② $(\forall n \in S)\,(P(n) \Rightarrow P(n+k))$

then $(\forall n \in S)\ P(n)$ holds.

---

e.g. if $S = \{2, 4, 6, \ldots\} = E$

and we can show

① $P(2)$

② if $P(n)$, then $P(n+2)$

then we've proved $P(n)$ holds $\forall n \in E$.

Ex: Consider the alternating sum of the first n squares

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1} n^2$$

$$= \sum_{k=1}^{n} (-1)^{k-1} k^2$$

Prop'n: ① if n is odd we have:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k \quad \left(= \frac{n(n+1)}{2} \text{ by our prev. proof}\right)$$

② if n is even, we have:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k \quad \left(= -\frac{n(n+1)}{2}\right)$$

Pf: here $n_0 = 1$, jump $= 2$, so $S = \{1, 3, 5, \ldots\}$

$P(n)$ is " $\sum_{k=1}^{n} (-1)^{k-1} k^2 = \sum_{k=1}^{n} k$ "

(BC) if $n=1$, $\sum_{k=1}^{1}(-1)^{k-1}k^2 = 1^2 = 1$
$= \sum_{k=1}^{1} k$ ✓

So P(1) holds.

(IH) Fix $n \in \{1,3,5,\ldots\}$ and assume P(n),

i.e. assume
$$\sum_{k=1}^{n}(-1)^{k-1}k^2 = \sum_{k=1}^{n} k$$

(IS) now consider the n+2 sum:
$$\sum_{k=1}^{n+2}(-1)^{k-1}k^2 = \left(\sum_{k=1}^{n}(-1)^{k-1}k^2\right) + (-1)^n(n+1)^2 + (-1)^{n+1}(n+2)^2$$

$$= \left(\sum_{k=1}^{n}(-1)^{k-1}k^2\right) - (n+1)^2 + (n+2)^2$$
(since n odd)

$$\overset{\text{IH}}{=} \sum_{k=1}^{n} k + [(n+2)^2 - (n+1)^2]$$

$$= \sum_{k=1}^{n} k + [\underbrace{(n+2) - (n+1)}_{1}][(n+2)+(n+1)]$$

$$= \sum_{k=1}^{n} k + (n+2) + (n+1)$$

$$= \sum_{k=1}^{n+2} k$$ ✓ So P(n+2) holds ✓

By induction we have true {1, 3, ...},
$$\sum_{k=1}^{n}(-1)^{k-1}k^2 = \sum_{k=1}^{n} k$$ ✓

Summary: we showed: ① P(1) holds

② if $n \in \{1,3,5,\dots\}$ then $P(n) \Rightarrow P(n+2)$

$\Rightarrow$ hence $P(n)$ holds $\forall n \in \{1,3,\dots\}$.

② For n even:

(BC) if $n=2$ we have:

$$\sum_{k=1}^{2} (-1)^{k-1} k^2 = 1^2 - 2^2 = -3 = -(1+2) = -\sum_{k=1}^{2} k \checkmark$$

(IH) Fix $n \in \{2,4,6,\dots\}$ and assume:

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = -\sum_{k=1}^{n} k$$

(IS) Now consider:

$$\sum_{k=1}^{n+2} (-1)^{k-1} k^2 = \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^n (n+1)^2 + (-1)^{n+1} (n+2)^2$$

$$= \sum_{k=1}^{n} (-1)^{k-1} k^2 + (n+1)^2 - (n+2)^2 \quad \text{(since } n \text{ even)}$$

IH

$$= -\sum_{k=1}^{n} (-1)^{k-1} k^2 \quad \big(\underbrace{(n+1) - (n+2)}_{-1}\big)\big((n+1) + (n+$$

$$= -\sum_{k=1}^{n} (-1)^{k-1} k^2 - [(n+1) + (n+2)]$$

$-(n+1) - n+2$

$$= -\sum_{k=1}^{n+2} (-1)^{k-1} k^2$$ ✓

by induction, the identity holds $\forall n \in \{2,4,6,\dots\}$

Fibonacci Sequence:

is defined recursively by:

$$f_0 = 0 \quad f_1 = 1$$

$$f_n = f_{n-2} + f_{n-1} \quad \text{for } n \geq 2.$$

$$\begin{array}{cccccccccc} 0, & 1, & 1, & 2, & 3, & 5, & 8, & 13, & 21, & \dots \\ f_0 & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & \dots \end{array}$$

↪ Fib sequence is a playground for inductive proofs.

Prop'n: $\forall n \in \mathbb{N}$ we have:

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

(i.e. $f_1 + f_2 + \dots + f_n = f_{n+2} - 1$)

Pf: (BC) if $n=1$, we have:

$$\sum_{k=1}^{1} f_k = f_1 = 1 = 2-1 = f_3 - 1 = f_{1+2} - 1 \checkmark$$

(IH) Fix $n \in \mathbb{N}$, assume

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1$$

(IS) Consider:

$$\sum_{k=1}^{n+1} f_k = \sum_{k=1}^{n} f_k + f_{n+1}$$

$$\overset{IH}{=} f_{n+2} - 1 + f_{n+1}$$

$$= f_{n+1} + f_{n+2} - 1$$

def'n of Fib seq $\overset{\text{}}{=} f_{n+3} - 1 = f_{(n+1)+2} - 1 \checkmark$

by PMI: $\forall n \in \mathbb{N}$ $\sum_{k=1}^{n} f_k = f_{n+2} - 1$ holds.

Prop'n If n is a multiple of 3 (i.e. if $n \in \{3,6,9,\ldots\}$) then f_n is even.

Pf: (BC) if $n=3$, then $f_n = f_3 = 4$, which is even $\checkmark$

(IH) Fix $n \in \{3,6,9,\ldots\}$. Assume f_n is even.

(IS) Consider f_{n+3}:

$$f_{n+3} = f_{n+2} + f_{n+1}$$

$$= (f_{n+1} + f_n) + f_{n+1}$$

$$= f_n + 2f_{n+1}$$

by the IH, f_n is even. Since $2f_{n+1}$ is even, $f_n + 2f_{n+1}$ is even, i.e. ~~f_{n+3}~~ is even.

By induction, $\forall n \in \{3, 6, 9, \ldots\}$, f_n is even ✓

<u>Strong</u> <u>induction</u>.

- In certain proofs may need to assume more than $P(n)$ to prove $P(n+1)$.
- e.g. may need to assume $P(n)$ and $P(n-1)$ ... or even $P(n)$ <u>and</u> $P(n-1)$ <u>and</u> ... <u>and</u> $P(1)$
- still a legit induction hypothesis!

<u>Thm</u> (Principle of strong mathematical induction (PSMI))

Sps $P(n)$ is a variable prop'n

If ① $P(1)$ holds, and
② $(\forall n \in \mathbb{N})\left[\left((\forall k \in [n]) P(k)\right) \Rightarrow P(n+1)\right]$ holds

then $(\forall n \in \mathbb{N}) P(n)$ holds.

"$\forall k \leq n$"

Template for a strong induction proof:

① Prove $P(1)$
② Fix $n \in \mathbb{N}$. Assume $(\forall k \in [n]) P(k)$
(i.e. assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$)
③ Deduce $P(n+1)$

→ PSMI then gives that $(\forall n \in \mathbb{N}) P(n)$ holds.

Side note: – despite name, PSMI seems weaker than PMI, because we have to assume more (all of $P(1) \wedge \cdots \wedge P(n)$ instead of just $P(n)$) to prove $P(n+1)$

– we'll later show PMI and PSMI are equivalent (and both equiv to another principle called WOP).

Ex's ① Let s_n be the sequence defined recursively by:

$$s_0 = 1$$
$$s_n = 1 + \sum_{k=0}^{n-1} s_n \quad \text{for } n \geq 1.$$

So e.g. $s_1 = 1 + s_0 = 1+1 = 2$

$s_2 = 1 + s_0 + s_1 = 1+1+2 = 4$

$s_3 = 1 + s_0 + s_1 + s_2 = 1+1+2+4 = 8.$

Looks like $s_n = 2^n$

Let's prove this: we'll need a strong inductive hypothesis.

Prop'n: $\forall n \in \mathbb{N} \cup \{0\}$ we have $s_n = 2^n$.

(BC) if $n=0$, $s_0 = 1 = 2^0$ ✓

(Strong IH) Fix $n \in \mathbb{N} \cup \{0\}$, and assume for every $k \in \{0, 1, \ldots, n\}$ we have

$$s_k = 2^k.$$

(IS) Now consider:

Thm For every $n \in \mathbb{N}$, $n > 1$, (i.e $\forall n \in \{2, 3, 4, \ldots\}$)
n has a prime factorization.

Pf: Let $P(n)$ be the prop'n
"n has a prime factorization"

(BC) $P(2)$ holds because 2 has a p.f.
(IH) Fix $n \in \{2, 3, 4, \ldots\}$ and assume for every $k \in \{2, 3, \ldots, n\}$ that $P(k)$ holds, i.e. k has a p.f.
(IS) Consider $n+1$. If $n+1$ is prime then $n+1 = n+1$ is a p.f. for $n+1$
If $n+1$ is not prime, then it can be factored:

$$n+1 = a \cdot b, \text{ where } a, b \in \{2, 3, \ldots, n\}$$

neither are 1, hence neither are $n+1$.

by the IH, a and b have p.f.'s

$$a = p_1 p_2 \cdots p_k$$
$$b = q_1 q_2 \cdots q_\ell$$

but then $n+1 = p_1 p_2 \cdots p_k q_1 q_2 \cdots q_\ell$ is a p.f.
i.e. $P(n+1)$ holds.
By (strong) induction, $\forall n \in \{2, 3, \ldots\}$ $P(n)$ holds, i.e. all $n \geq 2$ have p.f.'s.

(The treachery of...) <u>Multiple base cases</u>

→ sometimes need to check <u>more than one</u> base case in order to make a valid IH/IS.

- esp w/ recursively defined sequences.

<u>ex</u>: Define a seq x_n by:

$$x_1 = 2$$
$$x_2 = 3$$
$$x_n = 3x_{n-1} - 2x_{n-2} \qquad n \geq 3$$

<u>Prop'n</u>: $(\forall n \in \mathbb{N})(x_n = 2^{n-1} + 1)$

<u>Pf</u>: (BCs) if <u>$n=1$</u>: $x_1 = 2 = 2^{1-1}+1$ ✓

<u>$n=2$</u>: $x_2 = 3 = 2^{2-1}+1$ ✓

(IH) Fix $n \geq 2$ and assume $\forall k \in \{1, 2, .., n\}$ that $x_k = 2^{k-1}+1$.

(<u>notice</u>: IH should always fix $n \geq$ last base case verified ... that's our "springoffer point).

(IS) Then $x_{n+1} \overset{\text{def'n of sequence}}{=} 3x_n - 2x_{n-1}$ $(*)$

$$\overset{IH}{=} 3(2^{n-1}+1) - 2(2^{n-2}+1)$$

$$= 3\cdot 2^{n-1} + 3 - 2^{n-1} - 2$$

$$= (3-1)2^{n-1} + 1$$

$$= 2\cdot 2^{n-1} + 1 = 2^n + 1$$

$$= 2^{(n+1)-1} + 1. \checkmark$$

By induction, the identity $x_n = 2^{n-1}+1$ holds for all $n \in \mathbb{N}$. ✓

---

note:- we really needed to check both $n=1,2$ as BCs.

- if we only checked $n=1$ and let our IH be:

"Fix $n \geq 1$ and assume $\forall k \in \{1,2,\dots,n\}$ we have $x_k = 2^{k-1}+1$."

then step $(*)$ would have been unjustified for $n=1$. In this case $*$ would be:

$$x_{1+1} = 3x_1 - 2x_0 \text{ undefined!!}$$

→ can cook up false induction proofs that play on this issue.

"Prop'n" Let x_n be defined as above. Then $\forall n \in \mathbb{N}$ we have:

$$x_n = 2^{n+1} - 2.$$

"Pf" (BC) if $n=1$, then

$$x_1 = 2 = 2^{1+1} - 2 \checkmark$$

(IH) Fix $n \geq 1$ and assume $\forall k \in \{1, 2, \ldots, n\}$ that $x_k = 2^{k+1} - 2$.

(IS) Then: $x_{n+1} = 3x_n - 2x_{n-1}$ (*)

$$\stackrel{IH}{=} 3(2^{n+1} - 2) - 2(2^n - 2)$$

$$= 3 \cdot 2^{n+1} - 6 - \cancel{2 \cdot 2^n}^{\,2^{n+1}} + 4$$

$$= 2 \cdot 2^{n+1} - 2$$

$$= 2^{n+2} - 2 = 2^{(n+1)+1} - 2 \checkmark$$

By induction identity is "proved" $\forall n \in \mathbb{N}$.

- of course, we can verify identity is wrong even for $n=2$

$$x_2 = 3 \neq 6 = 2^{2+1} - 2.$$

- the issue is exactly that (*) is not justified when $n=1$, but in our IH we're allowing possibility of $n=1$, since we're only verified $n=1$ in our BC.

PMI, PSMI, and WOP

"Theorem" (well-ordering principle (WOP))
If $X \subseteq \mathbb{N}$ and $X \neq \emptyset$ then X has a least element (i.e. $(\exists x \in X)(\forall y \in X)(x \leq y)$)

- this "theorem" is intuitively obvious for $\mathbb{N}$. and is often taken as an axiom
- for example:
  - if $X = \mathbb{N}$, then X's least el't is 1.
  - if $X = \{2, 4, 6, \ldots\}$ then X's least el't is 2.
  - if $X = \{n \in \mathbb{N} \mid (\exists k \in \mathbb{N})(k > 5)(n = k^2)\}$
    $= \{36, 49, 64, 81, \ldots\}$
    then X's least el't is 36.
- though it's obvious, we can prove WOP using strong induction.

PF: we want to prove:
$(\forall X \in P(\mathbb{N}))\ (X \neq \emptyset \Rightarrow X$ has a least el't$)$

- So... Fix $X \in P(\mathbb{N})$ (i.e. $X \subseteq \mathbb{N}$). WTS: $X \neq \emptyset \Rightarrow X$ has least el't
- we argue by contrapositive:
  - ↳ assume X has no least el't.
  - ↳ we'll prove $X = \emptyset$ by strong induction.
- More specifically we'll prove by induction $(\forall n \in \mathbb{N})\ \underbrace{n \notin X}_{\text{call this } P(n)}$.

- (BC) $P(1)$ is true (i.e. $1 \notin X$), because if $1 \in X$ then it would be least el't of X (it is least el't of $\mathbb{N}$!)
- (IH) Fix $n \in \mathbb{N}$. Assume $\forall k \in \{1, 2, \dots, n\}$ we have $k \notin X$.
- (IS) Consider $n+1$. If $n+1 \in X$ then it would be least in X, since by IH $1 \notin X \wedge 2 \notin X \wedge \dots \wedge n \notin X$.
- Hence $n+1 \notin X$, since X has no least el't.
- i.e. $P(n+1)$ holds.

by strong induction, $(\forall n \in \mathbb{N})\ P(n)$ holds
i.e. $(\forall n \in \mathbb{N})\ n \notin X$ holds
i.e. $X = \emptyset$.

Since $X \in P(\mathbb{N})$ was arbitrary, we've proved WOP. ✓

We just showed:

PSMI $\Rightarrow$ WOP.

In fact, PSMI and WOP are equivalent (i.e. we can also prove WOP $\Rightarrow$ PSMI) and more: both are equivalent to PMI.

Thm The following are equivalent (TFAE):

1. PMI
2. PSMI
3. WOP

i.e. PMI $\Leftrightarrow$ PSMI $\Leftrightarrow$ WOP

i.e. from any one of these statements

i.e. you can prove the other two.

Pf: we've already shown:

PSMI $\Rightarrow$ WOP

Hence if we can show:

Claim 1: PMI $\Rightarrow$ PSMI

and Claim 2: WOP $\Rightarrow$ PMI

we will have proved the equivalence of the three statements:

$$\begin{tikzcd} & \text{PMI} \arrow[dr, Rightarrow, "\text{Claim 1}"] & \\ \text{WOP} \arrow[ur, Rightarrow, "\text{Claim 2}"] & & \text{PSMI} \arrow[ll, Rightarrow, "\text{know}"] \end{tikzcd}$$

Proof of PMI $\Rightarrow$ PSMI.

Assume PMI. i.e. assume: for every prop'n $P(n)$, if $P(1)$ and $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ both hold then $(\forall n \in \mathbb{N})\, P(n)$ holds.

Want to prove PSMI: i.e. wtp for every prop'n $Q(n)$, if $\overset{(a)}{Q(1)}$ and $\overset{(b)}{(\forall n \in \mathbb{N})}((\forall k \in [n])\, Q(k)) \Rightarrow Q(n+1))$ both hold, then $(\forall n \in \mathbb{N})\,(Q(n))$ holds.

- So fix a prop'n $Q(n)$, and assume (a) and (b) both hold.
- Let $P(n)$ be the prop'n "$(\forall k \in [n])\, Q(k)$"
- then: (BC) $P(1)$ holds since this is just $(\forall k \in [1])\, Q(k)$, which is equiv to $Q(1)$, which we knew holds by our assumption (a).
- (IH) Fix $n \in \mathbb{N}$ and assume $P(n)$ holds, i.e. assume $(\forall k \in [n])\, Q(k)$ holds
- then by (b) we know $Q(n+1)$ holds hence: $(\forall k \in [n])\, Q(k)$ and $Q(n+1)$ hold
- hence $(\forall k \in [n+1])\, Q(k)$ holds
- i.e. $P(n+1)$ holds.

Proof of WOP $\Rightarrow$ PMI

Assume WOP, and sps $P(n)$ is a var. prop'n

Sps we knew

① $P(1)$ holds

② $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(n+1))$ holds

- We want to show: $(\forall n \in \mathbb{N})\, P(n)$ holds.
- Let $S = \{n \in \mathbb{N} \mid P(n) \text{ fails}\}$
- we'll use WOP to prove $S = \emptyset$.
- if $S \neq \emptyset$ then by WOP S has a least el't x.
- we know $x \neq 1$, since $P(1)$ holds,
- hence $x = n+1$ for some $n \in \mathbb{N}$.
- x is least number for which P fails – so $P(n)$ holds
- but then by ②, $P(n+1)$ holds, i.e. $P(x)$ holds, a contradiction
- hence $S = \emptyset$, i.e. $(\forall n \in \mathbb{N})$ we have $P(n)$ ✓

↳ PMI, PSMI, WOP are all intuitively obvious principles, and often taken as axioms:

↳ Thm just says: if you assume any one of them, you can prove the other two.