Chapter 4: Intro to Mathematical Logic.

Goals: learn how to write statements more formally (w/ more symbols, fewer words)

- see how: the form of a statement suggests the form of its proof.

Recall: Def'n (intuitive): A mathematical statement (or prop'n) is a grammatically correct declarative sentence that is true or false

↳ may consist of words and/or symbols

- "statement" can be rigorously defined, but need more formal logic
- in that context, "grammatically correct" also has precise meaning.

Ex's ① Every integer is a real number (T)

② Every real number is an integer (F)

③ There exists $x \in \mathbb{R}$ s.t. $x \notin \mathbb{Z}$ (T). ②

④ $1+1=2$ (T)

⑤ There are infinitely many twin primes (unknown ... but either T/F).

Nonex's ① $\{!\pi$ (grammatically incorrect/ meaningless)

② Shakespeare (not declarative/ no truth value)

③ $x^2+1=2$

↳ meaningful sequence of symbols asserting an equality ... but no truth value unless x is specified.

- called a variable proposition: a sentence that becomes a statement once its variables are specified (or quantified over ... more on this later).

- we'll use $P, Q, R, \ldots$ for statements and $P(x), Q(x,y), \ldots$ for var. prop'ns.

e.g. might say: - Let P denote "$5^2+1=2$" (F)

- Let $Q(x)$ denote "$x^2+1=2$"

Then $Q(5)$ is the statement $5^2+1=2$ (F)

and $Q(1)$ is $1^2+1=2$ (T)

Mere var. prop'ns : ① $x^2+1 \leq 0$

② $x \in \mathbb{Z}$ and $x^2 < 39$

③ $z = x+y$.

indicate when abbreviating a var. prop'n w/ multiple variables, e.g. could use $Q(x,y,z)$ to denote ③.

Then: $Q(1,2,3)$ is F

but $Q(5,2,3)$ is T.

Quantifiers: the other way to turn a var. prop'n into a statement is to quantify over its variables.

e.g. "$x^2+1=2$" is a var. prop'n

but "There exists $x \in \mathbb{R}$ s.t. $x^2+1=2$" is a statement (T)

as is: "For every $x \in \mathbb{R}$ we have $x^2+1=2$" (F)

The clauses "There exists $x \in S$..."

and "For every $x \in S$..."

are two types of quantification of the variable x.

- We'll use the symbols:

$\forall$ read: "for all" or "for every"

$\exists$ read: "there exists"

called the "universal quantifier"

called the "existential quantifier"

- Given a var prop'n $P(x)$ and a set S, we have that:

"For all $x \in S$ we have $P(x)$"

"there exists $x \in S$ such that $P(x)$"

are <u>statements</u>.

- we denote these by:

$(\forall x \in S)\ P(x)$

$(\exists x \in S)\ P(x)$

(Book uses: $\forall x \in S \,.\, P(x)$

$\exists x \in S \,.\, P(x)$)

respectively.

<u>Ex's</u>:

① $(\exists x \in \mathbb{N})\ (x < 5)$

read: "there exists $x \in \mathbb{N}$ s.t. $x < 5$" (T)

② $(\forall x \in \mathbb{N})\ (x < 5)$

"For every $x \in \mathbb{N}$, we have $x < 5$" (F).

③ $(\forall x \in \mathbb{N})\ (x > 0)$ (T)

④ $(\forall x \in \mathbb{R})\ (x > 0)$ (F)

Multiple quantifiers:

5) $(\forall x, y \in \mathbb{N})(x+y \geq 2)$

read: "For all x and y in $\mathbb{N}$, we have $x+y \geq 2$" (T)

- can also nest $\forall$'s and $\exists$'s but beware: order of quantifiers is important!

6) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(y^2 = x)$

"For every $x \in \mathbb{N}$ there is $y \in \mathbb{R}$ s.t. $y^2 = x$"

i.e. every natural number has a real square root (T)

7) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

i.e. every natural # has a square root in $\mathbb{N}$ (F).

- what happens if we reverse the order of quantifiers in 6?

get:

$(\exists y \in \mathbb{R})(\forall x \in \mathbb{N})(y^2 = x)$

i.e. "there is a real number y s.t. every natural number is equal to y^2"

- perfectly well-written statement, but absurd and definitely <u>false</u>
- <u>moral</u>: order of quantifiers makes a big diff!

↳ can also have "inside quantifiers" e.g.

8 $(\forall x \in \mathbb{N})(x > 0 \text{ and } (\exists y \in \mathbb{N})(y > x))$

9 $(\forall x \in \mathbb{R})(\text{if } x \geq 0, \text{ then } (\exists y \in \mathbb{R})(y^2 = x))$

are both statements (both T).

<u>Note</u> on quantifying set variables:

- we've insisted all quantified variables range over a specific set, e.g. $(\forall x \in \mathbb{R})(x^2 \geq 0)$ is meaningful

$(\forall x)(x^2 \geq 0)$ is <u>not</u>.

- what if we want to quantify over variables referring to <u>sets</u>?

- e.g. to write:

"For every set S, we have $\emptyset \subseteq S$"

symbolically, might try:

$(\forall S \in (\ldots))(\emptyset \subseteq S)$

set of all sets??

- but the collection of all sets is not a set (Russell's paradox)
- convention: when quantifying set variables we'll write sentences verbally:

  i.e. "For every set S, ..."

  "For all sets $A, B, \ldots$"

Connectives and Truth Tables

- Connectives are symbols used to combine multiple statements into one.
- all our connectives will be binary (combine two statements into one) except negation which is unary.
- Truth tables tell us how truth of connected statements depends on truth of the original constituent statements

Conjunction ("and")

- conjunction of statements P, Q is written $P \wedge Q$ ("P and Q")
- $P \wedge Q$ is true iff both P, Q true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex's: Let P denote:

$$(\forall x \in \mathbb{Z})(x+1 > x)$$

Let Q be:

97 is prime

Let R be:

$$2^2 = 5.$$

Then P, Q are (T) but R is (F).

Hence $P \wedge Q$ is (T)

but $P \wedge R$ and $Q \wedge R$ are both (F).

- written out, $P \wedge Q$ is:

$$(\forall x \in \mathbb{Z})(x+1 > x) \wedge (97 \text{ is prime})$$

inserting parentheses can clarify an expression.

Disjunction ("or")

- disjunction of P, Q written $P \vee Q$ ("P or Q")
- $P \vee Q$ is true iff at least one of P, Q true:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

e.g.: $\overset{T}{(\forall x \in \mathbb{R})(x^2 \geq 0)} \vee \overset{F}{(96 \text{ is prime})}$

is (T), but

$(\forall x \in \mathbb{R})(x \geq 0) \vee (96 \text{ is prime})$

is (F).

Negation ("not")

- only unary connective we'll use
- negation of P written $\neg P$
- $\neg P$ true iff P is false:

P	$\neg P$
T	F
F	T

Ex's: ① $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is (F), hence:

② $\neg(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is (T), hence:

③ $\neg\neg(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$

is (F) again.

④ for <u>any</u> statement P, the statement $P \vee \neg P$ is (T), whereas $P \wedge \neg P$ is (F).

e.g. (96 is prime) $\vee \neg$(96 is prime) is (T)
but (96 is prime) $\wedge \neg$(96 is prime) is (F)

↳ can use connectives in <u>var.</u> <u>prop'ns</u> too

e.g. let $P(x,y)$ denote:
$(x>0) \wedge (y$ is prime$)$

then $P(3,5)$ is true
while $P(3,6)$ is false.
and $(\exists x,y \in \mathbb{N})\, P(x,y)$ is true
while $(\forall x,y \in \mathbb{N})\, P(x,y)$ is false.

↳ can also use in def'ns, set-builder notation etc.

- e.g. if A, B are subsets of a universal set U then..

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$
$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$
$$\bar{A} = \{x \in U \mid \neg(x \in A)\}$$

- we'll explore connection between set operations and connectives more later.

Implication:

- Given statements P, Q the statement $P \Rightarrow Q$ is read "P implies Q" or "if P, then Q."
  - $P \Rightarrow Q$ is true if whenever P is true, Q is also true.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- notice: $P \Rightarrow Q$ is always true when P is false, which is often confusing when first learning this connective
- $P \Rightarrow Q$ is only (F) when P is (T) but Q is (F).

- statements of form $P \Rightarrow Q$ are called <u>conditional</u> <u>statements</u>.

<u>Ex's</u>:

1. " $\overset{\text{or } T}{1+1=2} \Rightarrow \overset{F}{(1+1)+1=3}$ " is <u>true</u>
2. " $1+1=2 \Rightarrow (1+1)+1=4$ " is <u>false</u>
3. " $1+1=2 \Rightarrow \sqrt{2} \notin \mathbb{N}$ " is <u>true</u>

even though the P and Q in this example are not apparently related statements.

4. "My name is Sally $\Rightarrow$ my name begins with S." is <u>true</u>.
   - both the premise P and conclusion Q in this case are (F), but the conditional $P \Rightarrow Q$ is true.
   - illustration "false implies false" ~~[illegible]~~ is true.

5. "Tomorrow is Sunday $\Rightarrow$ my name is Garrett" is also <u>true</u>.
   (" false $\Rightarrow$ true " is true)

6. $(\exists x \in \mathbb{R})(x^2 = -1) \Rightarrow 1+1=3$ is <u>true</u>! Automatically since premise $(\exists x \in \mathbb{R})(x^2=-1)$ is <u>false</u>, despite fact it has no apparent relation to conclusion.

(7) can also use $\Rightarrow$ in var prop'ns

e.g.

$$x \geq 2 \Rightarrow x^2 \geq 4$$

is a well-formed var. prop'n and

$$(\forall x \in \mathbb{R})\ (x \geq 2 \Rightarrow x^2 \geq 4)$$

is <u>true</u>, because:

for every $x \in \mathbb{R}$, <u>either</u> $x \geq 2$, in which case $x^2 \geq 4$ (i.e. $x \geq 2 \Rightarrow x^2 \geq 4$ holds because "true $\Rightarrow$ true" is <u>true</u>); <u>or</u> $x < 2$ in which case $x \geq 2 \Rightarrow x^2 \geq 4$ holds automatically (because "false $\Rightarrow \ldots$" is <u>true</u>).

(8) <u>OTOH</u>: $(\forall x \in \mathbb{R})\ (x^2 \geq 4 \Rightarrow x \geq 2)$ is <u>false</u> because there is a real number x (e.g. $x = -3$) such that

$x^2 \geq 4$ is true, <u>but</u>
$x \geq 2$ is false.

Equivalence

Given statements P, Q the statement $P \Leftrightarrow Q$ (read: "P if and only if Q"
or: "P iff Q")

is true if and only if P, Q have the same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

ex's: ① $(1+1=2) \Leftrightarrow (1+1+1=3)$ is (T)

② $(1+1=3) \Leftrightarrow (1+1+1=4)$ is (T)

P, Q need not be related! → ③ $(\forall x \in \mathbb{N})(n>0) \Leftrightarrow (1+1=2)$ is (T)

④ $(1+1=2) \Leftrightarrow (2+2=5)$ is (F)

⑤ $(\forall x \in \mathbb{R})(x \geq 0 \Leftrightarrow (\exists y \in \mathbb{R})(x=y^2))$ is (T)

↓

why: For any fixed x_0 in $\mathbb{R}$, the statements "$x_0 \geq 0$" and "$(\exists y \in \mathbb{R})(y^2 = x_0)$" are either: both true, or both false.

Def'n Two statements P, Q are said to be logically equivalent, if $P \Leftrightarrow Q$ is true.

- e.g. $1+1=2$ and $1+1+1=3$ are logically equiv.
- we're more interested in logically equiv. forms for connected (esp. negated) and quantified statements.

Negating Quantified Statements:

- Sps $P(x)$ is a var. prop'n, and S is a set.
- Consider the negated statements:

① $\neg (\forall x \in S) P(x)$

② $\neg (\exists x \in S) P(x)$

observe: ① is true iff there is an $x \in S$ s.t. $P(x)$ is false, i.e. iff $(\exists x \in S) \neg P(x)$ is true.

② is true iff for every $x \in S$ we have $P(x)$ is false, i.e. iff $(\forall x \in S) \neg P(x)$ is true.

this shows:

$$\neg(\forall x\in S)\,P(x) \iff (\exists x\in S)\,\neg P(x)$$

is always <u>true</u> (regardless of the prop'n $P(x)$)
i.e. that $\neg(\forall x\in S)\,P(x)$ and $(\exists x\in S)\,\neg P(x)$ are <u>logically equiv</u>.

likewise:

$$\neg(\exists x\in S)\,P(x) \iff (\forall x\in S)\,\neg P(x)$$

is always <u>true</u>.

$\rightarrow$ these equivalences are often useful when trying to prove quantified statements by contradiction.

<u>Ex's</u>: ① $\neg(\forall x\in\mathbb{R})(x\in\mathbb{N})$ "not all reals are naturals"

is equiv. to:
$(\exists x\in\mathbb{R})\,\neg(x\in\mathbb{N})$ "there is a real which is not a natural"

(note: we'll often write $\neg(x\in\mathbb{N})$ as $x\notin\mathbb{N}$, $\neg(x=y)$ as $x\neq y$, etc...)

② $\neg(\exists x\in\mathbb{R})(x+1=0)$ "there is no additive inverse to 1 in $\mathbb{R}$"

is equiv. to
$(\forall x\in\mathbb{R})(x+1\neq 0)$ "every real is not an additive inverse for 1."

(in this case: both statements are <u>false</u>).

③ For multiple quantifiers, just iterate the process...

$$\neg(\forall x\in\mathbb{R})(\exists y\in\mathbb{R})(xy=1)$$ → "not every real has a multiplicative inverse"

is equiv to: $(\exists x\in\mathbb{R})\neg(\exists y\in\mathbb{R})(xy=1)$

is equiv to: $(\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(xy\neq 1)$ → "there is a real w/o a multiplicative inverse"

in this case: all are true since $x=0$ has no inverse.

Negating connected statements

Prop'n For any statements P, Q the following logical equivalences hold (i.e. the following statements are always true).

① $\neg\neg P \Leftrightarrow P$

② $\neg(P\wedge Q)\Leftrightarrow \neg P\vee\neg Q$

③ $\neg(P\vee Q)\Leftrightarrow \neg P\wedge\neg Q$

Pf: to prove, we'll use truth tables.

①

P	$\neg P$	$\neg\neg P$	$\neg\neg P\Leftrightarrow P$
T	F	T	T
F	T	F	T

← always true