↳ main import of def'n is in proofs: to prove $A = B$ one proves

(i) $A \subseteq B$

(ii) $B \subseteq A$.

↳ more on this later...

Powersets

- Consider the set $A = \{1, 2, 3\}$
- can we list all subsets of A?

yes:

$\emptyset$ $\{1\}$, $\{2\}$, $\{3\}$ $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ $\{1,2,3\}$

- the set of all these subsets is the <u>powerset</u> of A:

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

<u>Def'n</u> Given a set X, the <u>powerset of X</u>, denoted $P(X)$, is the set of all subsets of X, i.e. $Y \in P(X)$ iff $Y \subseteq X$

Ex: ① - $\{1, 27, 10^6\} \subseteq \mathbb{N}$, hence $\{1, 27, 10^6\} \in P(\mathbb{N})$

- Also, $E \in P(\mathbb{N})$, $O \in P(\mathbb{N})$, $\mathbb{N} \in P(\mathbb{N})$
- but $\{-1, 0, 1\} \notin P(\mathbb{N})$ since $\{-1, 0, 1\} \not\subseteq \mathbb{N}$.

② Prop'n: For any set X we have:

(i) $\emptyset \in P(X)$
(ii) $X \in P(X)$

Pf: we knew $\emptyset \subseteq X$ and $X \subseteq X$ ✓

③ Prop'n For any sets A, B, if $A \subseteq B$ then $P(A) \subseteq P(B)$.

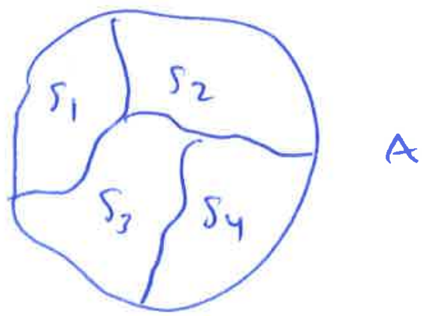
Pf: - Fix an $X \in P(A)$
- then $X \subseteq A$, by def'n of powerset.
- Since $A \subseteq B$, we have $X \subseteq B$ by transitivity of $\subseteq$ (proved before)
- hence $X \in P(B)$
- Since X was arbitrary, we have $P(A) \subseteq P(B)$. ✓

④ - What is $P(\emptyset)$?
- only subset of $\emptyset$ is $\emptyset$.
- So: $P(\emptyset) = \{\emptyset\}$

Partitions

A partition of a set A is a collection of subsets that split up A into disjoint pieces:

Picture:



partition of A into 4 pieces.

Formal def'n: Let A be a set. A partition of A is a collection (i.e. set) P of sets s.t.

(i) for every $X \in P$, we have $X \subseteq A$ and $X \neq \phi$.

(ii) for every $X, Y \in P$, if $X \neq Y$ then $X \cap Y = \phi$.

(iii) $\bigcup_{X \in P} X = A$.

Vocab: (ii) says that the sets $X \in P$ are "pairwise disjoint"

Ex's ① Let $A = \{1,2,3,4,5,6\} = [6]$.

Let $S_1 = \{1\}$

$S_2 = \{2,3,6\}$

$S_3 = \{4,5\}$.

Then $P = \{S_1, S_2, S_3\}$ is a partition of A.

$= \{\{1\}, \{2,3,6\}, \{4,5\}\}$

Why: (i) S_1, S_2, S_3 all nonempty subsets of A. ✓

(ii) $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = \emptyset$ ✓

(iii) $S_1 \cup S_2 \cup S_3 = A$ ✓

However: – $\{S_1, S_2\}$ is not a partition since (iii) fails.

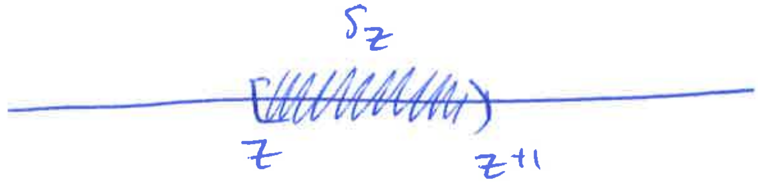
– Let $S_4 = \{1,6\}$. Then $\{S_1, S_2, S_3, S_4\}$ isn't a partition either, since (ii) fails (e.g. $S_1 \cap S_4 = \{1\} \neq \emptyset$).

② $\{E, O\}$ is a partition of $\mathbb{N}$.

③ – Consider $\mathbb{R}$.

– for every $z \in \mathbb{Z}$, define

$$S_z = \{x \in \mathbb{R} \mid z \le x < z+1\}$$
$$= [z, z+1)$$



e.g. $S_1 = [1,2)$.

Then $\{S_z : z \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$.
(why?)

4. OTOH ("on the other hand") if we define:

$$T_z = \{x \in \mathbb{R} \mid z \le x \le z+1\}$$
$$= [z, z+1]$$

then $\{T_z : z \in \mathbb{Z}\}$ is <u>not</u> a partition of $\mathbb{R}$, since e.g. $T_1 \cap T_2 = [1,2] \cap [2,3]$
$$= \{2\}$$
$$\ne \emptyset.$$

(Note: in general, if for every $i \in I$ we've defined A_i, I'll write $\{A_i : i \in I\}$ for the <u>set of sets</u> A_i indexed by I).

<u>Cartesian Products</u>

<u>Def'n</u>: Suppose A, B are sets. The <u>Cartesian Product</u> of A, B, denoted $A \times B$, is the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$

$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

illegal use of set-builder notation ... but can be formalized

<u>Note</u>: order is important! First coord. from A, second from B, and often $A \neq B$.

- sometimes write A^2 for $A \times A$.

<u>Ex's</u>: ① if $A = \{1,2,3\}$
$B = \{*, \heartsuit\}$.

then $A \times B = \{(1,*), (2,*), (3,*), (1,\heartsuit), (2,\heartsuit), (3,\heartsuit)\}$.

② $(1,2) \in \mathbb{N} \times \mathbb{N}$ and $(1,2) \in \mathbb{N} \times \mathbb{R}$
also $(1,\sqrt{2}) \in \mathbb{N} \times \mathbb{R}$ but $(1,\sqrt{2}) \notin \mathbb{N} \times \mathbb{N}$
since $\sqrt{2} \notin \mathbb{N}$.

③ <u>Triples</u>: Given sets A, B, C can define
$A \times B \times C = \{(a,b,c) \mid a \in A, b \in B, c \in C\}$
e.g. $(1, \pi, 2+i) \in \mathbb{N} \times \mathbb{R} \times \mathbb{C}$.

→ $A \times B \times C$ not same as $(A \times B) \times C$:
el'ts of first set look like (a,b,c)
and el'ts of second like $((a,b),c)$

→ but these sets "essentially" the same...

4. Can use pairs, triples, etc. in set-builder notation,

e.g. let $X = \{(m,n) \in \mathbb{N} \times \mathbb{N} \mid m \in E \text{ and } n \in O\}$

then $(2,5) \in X$

but $(1,6) \notin X$ nor is $(2,6)$.

Proofs with Sets.

Proving $A \subseteq B$:
1. Fix an arbitrary $a \in A$
2. Prove $a \in B$
3. Conclude, since a was arbitrary, that for every $x \in A$ we have $x \in B$, i.e. $A \subseteq B$

Ex 1 **Prop'n** Suppose A, B, X are sets. if $X \subseteq A$ and $X \subseteq B$ then $X \subseteq A \cap B$.

Pf: - fix $x \in X$
- since $X \subseteq A$, we have $x \in A$
- since $X \subseteq B$, we have $x \in B$ too.
- Hence $x \in A \cap B$.
- Since $x \in X$ was arbitrary, we have for every $y \in X$ that $y \in A \cap B$, i.e. $X \subseteq A \cap B$ ✓

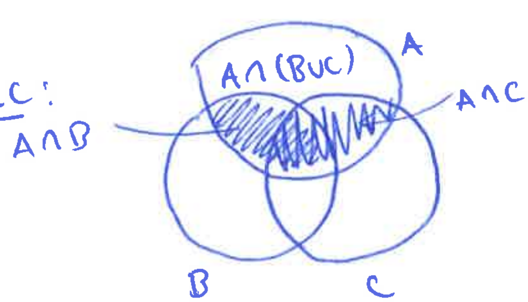
② **Prop'n**: Suppose A, B are sets.
Then $P(A) \cap P(B) \subseteq P(A \cap B)$.

Pf:
- Fix $X \in P(A) \cap P(B)$.
- then $X \in P(A)$ and $X \in P(B)$, by def'n of $\cap$.
- hence $X \subseteq A$ and $X \subseteq B$, by def'n of powerset.
- hence by our previous prop'n we have $X \subseteq A \cap B$, i.e. $X \in P(A \cap B)$
- Since X was arbitrary, we have $P(A) \cap P(B) \subseteq P(A \cap B)$.

Proving $A = B$:
1. Show $A \subseteq B$
2. Show $B \subseteq A$
3. Conclude $A = B$.

Ex: ① **Prop'n**: Let A, B, C be sets.
Then: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Pic:



$\mathcal{RC}$

Pf: ($\subseteq$) - fix $x \in A \cap (B \cup C)$

- then: (i) $x \in A$, and
  (ii) $x \in B \cup C$

↳ case 1: $x \in B$. Then since $x \in A$ as well we have $x \in A \cap B$ in this case.

↳ case 2: $x \notin B$. In this case we must have $x \in C$, since by (ii) $x \in B \cup C$ (i.e. $x \in B$ or $x \in C$).

Hence $x \in A \cap C$ in this case.

- Thus either $x \in A \cap B$ or $x \in A \cap C$.

i.e. $x \in (A \cap B) \cup (A \cap C)$

- Since x was arbitrary we have

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ✓

($\supseteq$) - fix $x \in (A \cap B) \cup (A \cap C)$

- case 1: $x \in A \cap B$.
  - then $x \in A$ and $x \in B$
  - hence $x \in A$ and ($x \in B$ or $x \in C$)
  - i.e. $x \in A \cap (B \cup C)$.

- case 2: $x \notin A \cap B$.
  - then since $x \in A \cap B$ or $x \in A \cap C$

must have $x \in A \cap C$.

- i.e. $x \in A$ and $x \in C$
- hence $x \in A$ and ($x \in B$ or $x \in C$)
- i.e. $x \in A \cap (B \cup C)$

↳ in either case we have $x \in A \cap (B \cup C)$

↳ since x was arbitrary, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

↳ hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ✓

Counterexamples: To show a statement is false, sufficient to provide a single counterexample.

Ex: Disprove the following:

Claim: For any sets A, B, C, if $A \subseteq B \cup C$ then either $A \subseteq B$ or $A \subseteq C$.

Sol'n: Consider the sets

$$A = \{2,3\}$$
$$B = \{1,2\}$$
$$C = \{3,4\}$$

then $A = \{2,3\} \subseteq \{1,2,3,4\} = B \cup C$

but $A \not\subseteq B$ and $A \not\subseteq C$, so the statement is false.

Using Contradiction

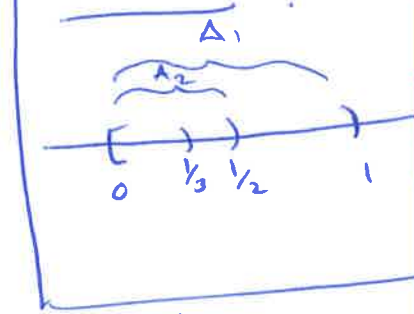
Strategy: to prove a statement S is true, you can:

1. Assume S is false
2. Show this assumption contradicts your other hypotheses (or a previously proved statement)
3. Conclude S is true.

Ex: **Prop'n**: For every $n \in \mathbb{N}$, define $A_n = \{x \in \mathbb{R} \mid 0 \leq x < \frac{1}{n}\} = [0, \frac{1}{n})$.

Then: $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$.

Picture:



Pf: ($\supseteq$) - Proving $\{0\} \subseteq \bigcap_{n \in \mathbb{N}} A_n$ is equivalent to proving $0 \in \bigcap_{n \in \mathbb{N}} A_n$, i.e. that $0 \in A_n$ for every $n \in \mathbb{N}$.

- Fix $n \in \mathbb{N}$. Then $A_n = [x \in \mathbb{R} \mid 0 \leq x < \frac{1}{n}\}$ so clearly $0 \in A_n$ since $0 \leq 0 < 1/n$ ✓
- Since n was arbitrary, we have $0 \in A_n$ for every n, i.e. $0 \in \bigcap_{n \in \mathbb{N}} A_n$ ✓

i.e. $\{0\} \subseteq \bigcap_{n \in \mathbb{N}} A_n$.

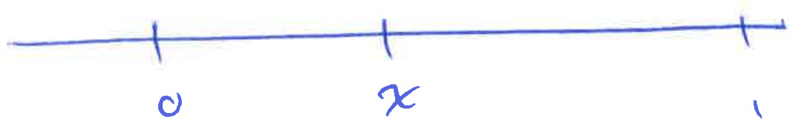
($\subseteq$) – To prove $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{0\}$ we argue by contradiction.

– Sps (toward a contradiction) that $\bigcap_{n\in\mathbb{N}} A_n \not\subseteq \{0\}$

– then there is $x \in \bigcap_{n\in\mathbb{N}} A_n$ s.t. $x \notin \{0\}$ i.e. $x \neq 0$.

– then in particular $x \in A_1 = [0,1)$

– then since $x \neq 0$ must have $0 < x < 1$.



– Pick an $N \in \mathbb{N}$ large enough so that $\frac{1}{N} < x$ (this is always possible – why?)

– but then $x \notin A_N$

– hence $x \notin \bigcap_{n\in\mathbb{N}} A_n$, a contradiction as we chose $x \in \bigcap_{n\in\mathbb{N}} A_n$.

- hence our assumption that $\bigcap_{n\in\mathbb{N}} A_n \not\subseteq \{0\}$ is false,

  i.e. $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{0\}$. ✓

- hence $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$ ✓