Hence $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$

i.e. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ✓

<u>Claim</u> Fix $n, k \in \mathbb{N}$, $k \leq n$. Then

$$\binom{n}{k} k = n \binom{n-1}{k-1}$$

<u>Pf</u>: Let S denote the set of committees of k people from a group of n people that include a specified chairperson.

Such a committee can be formed by:

- picking the committee members $\binom{n}{k}$
- from there, picking the chairperson $\binom{k}{1} = k$

hence $|S| = \binom{n}{k} k$

or could form a committee by: (2)

- picking a chairperson first $\binom{n}{1} = n$
- from remaining $n-1$ people pick remaining $k-1$ committee members $\binom{n-1}{k-1}$

hence $|S| = n\binom{n-1}{k-1}$ too

hence $n\binom{n-1}{k-1} = \binom{n}{k}k$. ✓

In this case, can verify the identity algebraically:

$$\binom{n}{k}k = \frac{n!}{k!(n-k)!}k = \frac{n!}{(k-1)!(n-k)!}$$

$$n\binom{n-1}{k-1} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$$

$$= \frac{n!}{(k-1)!(n-k)!}$$

same

<u>Prop'n</u>: Fix $n \in \mathbb{N}$. Then:

$$n\,2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

Pf: Let S be the set of nonempty committees w/ a chairperson chosen from a group of n people.

Done ✓

... But let's check:

can form committee by:

- choosing the chair $\binom{n}{1} = n$
- from remaining $n-1$, choosing other committee members

(i.e just choosing a subset from a set of size $n-1$)

$\Rightarrow |S| = n \cdot 2^{n-1}$

OTOH we can partition S:

$S = A_1 \cup A_2 \cup \cdots \cup A_n$

where A_K is the set of committees w/ exactly K people.

above we computed

$$|A_K| = \binom{n}{K} K$$

Hence
$$|S| = |A_1| + |A_2| + \cdots + |A_n|$$
$$= \binom{n}{1}1 + \binom{n}{2}2 + \cdots + \binom{n}{n}n$$
$$= \sum_{K=1}^{n} \binom{n}{K} K \checkmark$$

As a bonus, using previous example, could ~~re~~write this identity as:

$$n2^{n-1} = \sum_{K=1}^{n} \binom{n}{K} K = \sum_{K=1}^{n} n\binom{n-1}{K-1} \checkmark$$

<u>Inclusion</u> / <u>Exclusion</u>:

Ros says: if we write a set A

as:

$$A = A_1 \cup A_2 \cup \cdots \cup A_K$$

<u>and</u> then A_i's are disjoint, <u>then</u>

$$|A| = |A_1| + |A_2| + \dots + |A_k|.$$

... but what if the A_i's are <u>not</u> disjoint? Can we still count $|A|$ in terms of the $|A_i|$'s?

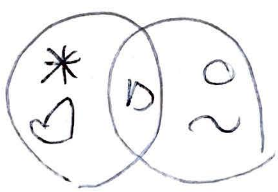
<u>Yes</u>: but need the principle of inclusion/exclusion.

<u>ex</u>: Let $A_1 = \{*, \heartsuit, \Delta\}$

$A_2 = \{\Delta, \circ, \sim\}$

Let $A = A_1 \cup A_2$

What is $|A|$?



A_1 A_2

is $|A| = |A_1| + |A_2|$? Not quite — since $\Delta \in A_1 \cap A_2$.

but we can think of counting $|A|$ as $|A_1| + |A_2|$ ... then correcting over-counting:

In this case we count the elements in $A_1 \cap A_2$ (i.e. Δ) twice, hence

$$\begin{aligned} |A| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= 3 + 3 - 1 \\ &= 5 \checkmark \end{aligned}$$

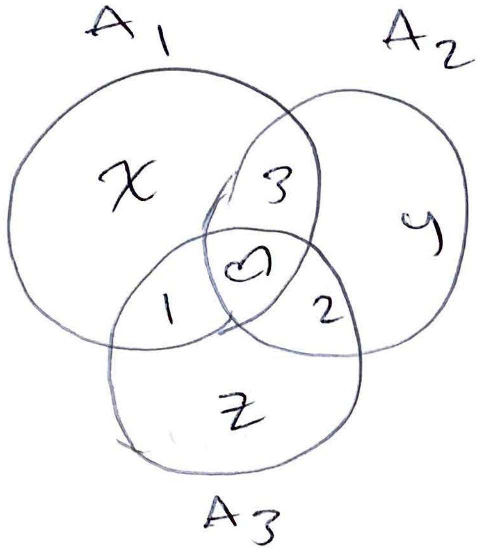
and indeed $A = A_1 \cup A_2 = \{*, \heartsuit, \Delta, \circ, \sim\}$

$\rightarrow$ this is general!

For any finite set A, if $A = A_1 \cup A_2$ then $|A| = |A_1| + |A_2| - |A_1 \cap A_2|$

what about 3 sets?

ex: sps
$$\begin{aligned} A_1 &= \{x, 1, 3, \heartsuit\} \\ A_2 &= \{y, 2, 3, \heartsuit\} \\ A_3 &= \{z, 1, 3, \heartsuit\} \end{aligned}$$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3|$$

el'ts here counted twice

$$- |A_2 \cap A_3|$$

$$+ |A_1 \cap A_2 \cap A_3|$$

el'ts here counted twice, then subtracted three times!

So need to add back!

Reasoning more generally like this yields:

<u>Theorem</u> (Principle of Inclusion/Exclusion) (PIE)

If A is finite and

$$A = A_1 \cup A_2 \cup \cdots \cup A_k$$

<u>then</u>: $|A| = |A_1 \cup A_2 \cup \cdots \cup A_k|$

$$= \sum_{i=1}^{k} |A_i| \quad \binom{k}{1} \text{ terms}$$

$$- \sum_{1 \le i < j \le k} |A_i \cap A_j| \quad \binom{k}{2} \text{ terms}$$

$$+ \sum_{1 \le i < j < l \le k} |A_i \cap A_j \cap A_l| \quad \binom{k}{3} \text{ terms}$$

$$-$$

$$\vdots$$

$$+ (-1)^{k-1} |A_1 \cap \cdots \cap A_k| \quad \binom{k}{k} \text{ terms}$$

Ex ① Sps in a group of 50 ppl

- 30 have brown hair
- 25 have blue eyes
- 10 have both

(i) How many have either brown hair or blue eyes (or both)?

(ii) How many have neither?

Sol'n: Let A_{brown} = set of brown hair

A_{blue} = set of blue eyes

then, by PIE:

(i) $|A_{brown} \cup A_{blue}| = |A_{brown}| + |A_{blue}|$

$- |A_{brown} \cap A_{blue}|$

$= 30 + 25 - 10$

$= 45$

(ii) So # w/ neither $= 50 - 45 = 5$

Ex ② How many integers between 1 and 1000 are not divisible by any of ~~5, 7, 11~~ the #'s 5, 7, 11? 29

Sol'n

Let A_k = set of integers between 1 and 1000 divisible by k

So $A_5 = \{5, 10, 15, \ldots, 1000\}$

$|A_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$

$A_7 = \{7, 14, \ldots, 994\}$

$|A_7| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$

and $|A_{11}| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$

Now: $A_5 \cap A_7$ = set of $n \le 1000$ divis by both 5, 7

i.e. divis by 35 $= \mathrm{lcm}(5,7)$

$= A_{35}$

hence $|A_5 \cap A_7| = |A_{35}|$

$$= \left\lfloor \frac{1000}{35} \right\rfloor = 28$$

Sim'ly: $|A_5 \cap A_{11}| = |A_{55}|$

$$= \left\lfloor \frac{1000}{55} \right\rfloor = 18$$

$|A_7 \cap A_{11}| = |A_{77}|$

$$= \left\lfloor \frac{1000}{77} \right\rfloor = 12$$

Finally $|A_5 \cap A_7 \cap A_{11}|$

$$= |A_{385}| = \left\lfloor \frac{1000}{385} \right\rfloor$$

$$= 2$$

So: by PIE:

$$|A_5 \cup A_7 \cup A_{11}| = |A_5| + |A_7| + |A_{11}|$$

$$- |A_5 \cap A_7|$$

$$- |A_5 \cap A_{11}| - |A_7 \cap A_{11}|$$

$$+ |A_5 \cap A_7 \cap A_{11}|$$

$= 200 + 142 + 90 - 28 - 18 - 12 + 2$

$= 376$

we're interested in the complement of this set

$\Rightarrow$ # of $n \leq 1000$ divisible by none of $5, 7, 11 = 1000 - 376$

$= 624$