Countin' Poker hands.

- a standard deck consists of 52 cards
- each card has 1 of 4 suits ($\heartsuit, \diamondsuit, \clubsuit, \spadesuit$)

  and 1 of 13 ranks: ($A, 2, 3, 4, \ldots, 9, 10, J, Q, K$)
- e.g. $A\diamondsuit$ and $2\heartsuit$ are cards
- a poker hand is a 5-selection from a standard deck.

Ex ① How many distinct hands are possible?

Sol'n $\binom{52}{5} = 2,598,960$.

② A full house is a hand consisting of 3 cards of one rank, and 2 cards of another, e.g. $A\diamondsuit, A\heartsuit, 3\spadesuit, 3\diamondsuit, 3\heartsuit$

Q: how many distinct full house hands are possible?

Sol'n

- pick two ranks $\binom{13}{2}$
- from these, pick the 3-card rank $\binom{2}{1}$
- pick three cards from this rank $\binom{4}{3}$
- pick two cards from the ~~###~~ 2-card rank $\binom{4}{2}$

$\Rightarrow$ # of full house hands is:

$$\binom{13}{2}\binom{2}{1}\binom{4}{3}\binom{4}{2} = 3{,}744.$$

3. A 3-of-a-kind consists of 3 cards from a single rank and 2 other cards from two other distinct ranks, e.g. 3 Q's, a 10, and a K.

Q: how many 3-of-a-kind hands are possible?

Sol'n: - pick 3-card rank $\binom{13}{1}$
- from this rank, pick 3 cards $\binom{4}{3}$

- Pick remaining two ranks $\binom{12}{2}$
- From the first of these, pick a card $\binom{4}{1}$
- and from the second $\binom{4}{1}$

$\Rightarrow$ # of 3-of-a-kinds is

$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 54,912$$

<u>Alt Sol'n</u>: - Pick three ranks $\binom{13}{3}$
- from these, pick the 3-card rank $\binom{3}{1}$
- from this rank, pick 3 cards $\binom{4}{3}$
- from other two ranks, pick cards $\binom{4}{1}\binom{4}{1}$

But: $\binom{13}{3}\binom{3}{1}\binom{4}{3}\binom{4}{1}\binom{4}{1} = 54,912$ too ✓

Binary sequences

- a binary sequence (of length n) is an ordered sequence of 0's and 1's (of length n)

- e.g. $s = 011$ is a binary sequence of length 3

- Let P_n denote set of all bin. seq's of length n.

Ex ① What is $|P_n|$?

② How many sequences $s \in P_n$ have at least two 1's?

Sol'n: ① each $s \in P_n$ formed by making n choices

$$\underbrace{\underline{0/1}\ \ \underline{0/1}\ \ \cdots\ \ \underline{0/1}}_{n}$$

2 choices each

$\Rightarrow |P_n| = 2 \cdot 2 \cdots 2$
$= 2^n$

② easier to count # of seq's w/ zero 1's or one 1 and subtract from total

\# w/ zero 1's $= 1$ (just $11\cdots1$)
\# w/ one 1 $= n$ (one for each place to put the 1)

so \# w/ at least two 1's

$= 2^n - n - 1$ ✓

<u>Theorem</u> Fix $n \in \mathbb{N} \cup \{0\}$

then: $2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

$= \sum_{k=0}^{n} \binom{n}{k}$

<u>Pf</u>: if $n = 0$, then $2^0 = 1 = \binom{n}{0} = \sum_{0}^{0} \binom{n}{k}$

So sps $n \geq 1$.
we proved $|P_n| = 2^n$
we can <u>partition</u> $P_n = S_0 \cup S_1 \cup \cdots \cup S_n$
where S_k = set of sequences w/ exactly k-many 1's.
<u>observe</u>: $|S_k| = \binom{n}{k}$ ← \# of ways to pick k positions where 1's appear.

Hence: $2^n = |P_n| = |S_0| + |S_1| + \cdots + |S_n|$
$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$
$= \sum_{k=0}^{n} \binom{n}{k}$ ✓

Selections w/ repetition:

Q: How many ways to select n objects from k types of objects, if repetition is allowed?

Ex: Dee's donuts sells 4 types of donuts, and you want to buy a dozen. How many distinct ways to do this?

Soln: Imagine putting down 3 "spacers" to separate donut types:

$$\underbrace{\circ\circ\circ}_{\text{type } 1} \mid \underbrace{\circ\circ}_{2} \mid \underbrace{\circ\circ\circ\circ\circ}_{3} \mid \underbrace{\circ\circ}_{4}$$

So the above "donut + spacer" diagram would correspond to a purchase of

3 donuts of type I
2 of type II
5 of type III
2 of type IV

$\rightsquigarrow$ can view such a diagram as a 01-sequence: w/ 12 0's (for a dozen donuts)
3 1's (to separate 4 types)

- Conversely any such sequence (12 0's 3 1's) corresponds to a selection of donuts.
- e.g. 101000000010000 corresponds to a selection of

$$\begin{array}{lll} 0 & \text{tp I} & \text{donuts} \\ 1 & \text{tp II} & \text{donuts} \\ 7 & \text{tp III} & \text{"} \\ 4 & \text{tp IV} & \text{"} \end{array}$$

- Hence: # of ways to make a selection = # of binary seqs of length 15 w/ three 1's

$= \binom{15}{3} = 455.$ ✓

Same reasoning in general proves:

Theorem The # of ways to select n objects from k types w/ repetition allowed is

$$\binom{n+(k-1)}{k-1}$$

(k-1 because we only need k-1 "spacers" to separate k types of objects)
ie

Ex: Sps we roll n (indistinguishable) 6-sided dice.
How many distinct outcomes are possible?

Sol'n: each of the n dice can roll into 6 possible "types"

1	2	3	4	5	6
oo	o		oooo		ooo

hence # of possible outcomes is:

$$\binom{n+(6-1)}{6-1} = \binom{n+5}{5}$$

So if we roll 10 dice # is:

$$\binom{15}{5} = 3003 \checkmark$$

Ex: How many anagrams of the word

LIMITING

are there?

Sol'n: Two approaches: ① First distinguish the I's w/ subscripts I_1, I_2, I_3

- # of anagrams w/ distinguished I's is just 8!
- For each anagram w/ distinguished I's there are 3! equivalent anagrams w/ I's not distinguished

(one for each permutation of $I_1 I_2 I_3$)

$\Rightarrow$ # of anagrams $= \dfrac{8!}{3!}$

(2) Can think of anagram being formed in two stages:

(i) pick 3 positions for the I's $\binom{8}{3}$

(ii) for remaining 5 positions pick an ordering of LMTNG: 5!

$\Rightarrow$ # of anagrams $= \binom{8}{3} \cdot 5!$

$$= \frac{8!}{3!5!} = \frac{8!}{3!}$$

$$= 6720, \text{ as before.}$$

Countin' in Two Ways:

Thm (Pascal's identity) Fix $n, k \in \mathbb{N}$ w/ $k \leq n$.

then:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pf: Let S = the set of k element subsets of $[n] = \{1, 2, \ldots, n\}$

Then $|S| = \binom{n}{k}$

OTOH we can partition S into S_1 and T where:

S_1 = k element subsets of $[n]$ that contain 1

T = k el't subsets of $[n]$ that do not contain 1.

Then $|S| = |S_1| + |T|$

- subsets in S_1 are formed by selecting $k-1$ elements from $\{2, 3, \ldots, n\}$ (1 automatically included)

$\Rightarrow |S_1| = \binom{n-1}{k-1}$

- subsets in T are formed by selecting k elements from $\{2, \ldots, n\}$

$\Rightarrow |T| = \binom{n-1}{k}$

Hence $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$

i.e. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ✓