

Counting Poker hands.

(10)

- a standard deck consists of 52 cards
- each card has 1 of 4 suits
($\heartsuit, \diamondsuit, \clubsuit, \spadesuit$)
and ~~one~~ 1 of 13 ranks:
(A, 2, 3, 4, ..., 9, 10, J, Q, K)
- e.g. $A \diamond$ and $2 \heartsuit$ are cards
- a poker hand is a 5-selection from a standard deck.

Ex ① How many distinct hands are possible?

Sol'n $\binom{52}{5} = 2,598,960$.

② A full house is a hand consisting of 3 cards of one rank, and ~~two~~ 2 cards of another, e.g. $A \diamond, A \heartsuit, 3 \clubsuit, 3 \diamond, 3 \heartsuit$

Q: how many distinct full house hands are possible?

Sol'n

- pick two ranks $\binom{13}{2}$ (10)

- from these, pick the 3-card rank $\binom{2}{1}$

- pick three cards from this rank $\binom{4}{3}$

- pick two cards from the ~~two~~ 2-card rank $\binom{4}{2}$

\Rightarrow # of FULL house hands is:

$$\binom{13}{2} \binom{2}{1} \binom{4}{3} \binom{4}{2} = 3,744$$

(3) A 3-of-a-kind consists of 3 cards from a single rank and 2 other cards from two other distinct ranks, e.g. 3 Q's, a 10, and a K.

Q: how many 3-of-a-kind hands are possible?

Sol'n: - pick 3-card rank $\binom{13}{1}$

- from this rank, pick 3 cards $\binom{4}{3}$

- Pick remaining two ranks $\binom{12}{2}$ ⑫
- From the first of these, pick a card $\binom{4}{1}$
- and from the second ~~rank~~ $\binom{4}{1}$

⇒ # of 3-of-a-kinds is

$$\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1} \binom{4}{1} = 54,912$$

- Alt Sol'n:
- Pick three ranks $\binom{13}{3}$
 - from these, pick the 3-card rank $\binom{3}{1}$
 - from this rank, pick 3 cards $\binom{4}{3}$
 - from other two ranks, pick cards $\binom{4}{1} \binom{4}{1}$

But: $\binom{13}{3} \binom{3}{1} \binom{4}{3} \binom{4}{1} \binom{4}{1} = 54,912$ too ✓

Binary Sequences

(13)

- a binary sequence (of length n) is an ordered sequence of 0's and 1's (of length n)

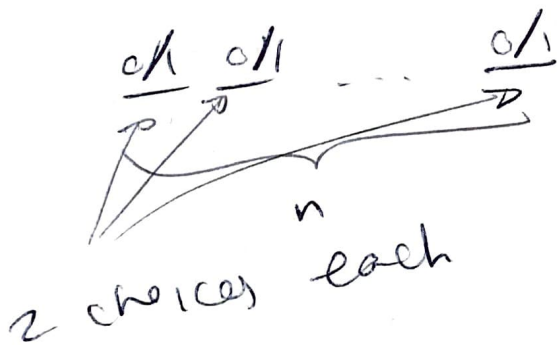
- e.g. $s = 011$ is a binary sequence of length 3.

- let P_n denote set of all bin. seqs of length n .

Ex ① what is $|P_n|$?

② How many sequences $s \in P_n$ have at least two 1's?

Sol'n: ① each $s \in P_n$ formed by making n choices



$$\Rightarrow |P_n| = 2 \cdot 2 \cdot \dots \cdot 2 \\ = 2^n$$

② easier to count # of seq's w/ zero 1's or one 1 and subtract from total.

w/ zero 1's = 1 (just 11...1)
 # w/ one 1 = n (one for each place to put the 1)

so # w/ at least two 1's
 = $2^n - n - 1$ ✓

Theorem Fix $n \in \mathbb{N} \setminus \{0\}$

then; $2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$
 $= \sum_{k=0}^n \binom{n}{k}$

PF: if $n=0$, then $2^0 = 1 = \binom{0}{0} = \sum_0^0 \binom{0}{k}$

So sps $n \geq 1$.

we proved $|P_n| = 2^n$

we can partition $P_n = S_0 \cup S_1 \cup \dots \cup S_n$
 where $S_k =$ set of sequences w/ exactly k -many 1's.

observe: $|S_k| = \binom{n}{k}$ ← # of ways to pick k positions when 1's appear.

Hence: $2^n = |P_n| = |S_0| + |S_1| + \dots + |S_n|$
 $= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$
 $= \sum_{k=0}^n \binom{n}{k}$ ✓

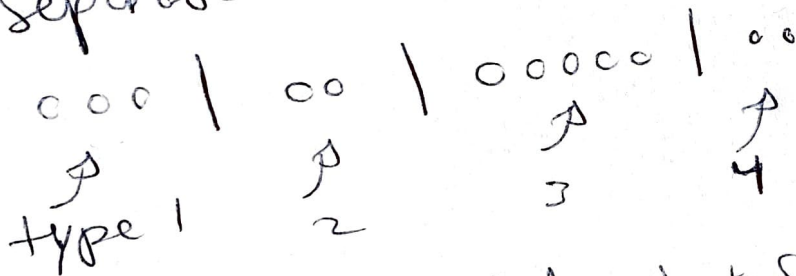
Selections w/ repetition:

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Q: How many ways to select n objects from k types of objects, if repetition is allowed?

Ex: Dee's donuts sells 4 types of donuts, and you want to buy a dozen. How many distinct ways to do this?

Sol'n: Imagine putting down 3 "spacers" to separate donut types:



So the above "donut + spacer" diagram would correspond to a purchase of 3 donuts of type I, 2 of type II, 5 of type III, and 2 of type IV.

↳ can view such a diagram as a 12-sequence: w/ 12 o's (for a dozen donuts) and 3 |'s (to separate 4 types)

- Conversely any such sequence (12 0's) (3 1's) (16)
 corresponds to a selection of donuts.

- e.g. 10100000010000
 corresponds to a selection of

0	tp I	donuts
1	tp II	donuts
7	tp III	"
4	tp IV	"

- Hence: # of ways to make a selection
 = # of binary seqs of length 15 w/
 three 1's
 = $\binom{15}{3} = 455$ ✓

Some reasoning in general proves:

Theorem The # of ways to select
~~n~~ n objects from k types w/
 repetition allowed is
 $\binom{n + (k-1)}{k-1}$

(k-1 because we only need k-1 "spacers" to separate k types of objects)

Ex: Spss we roll n (indistinguishable) (17)
6-sided dice.

How many distinct outcomes are possible?

Sol'n: each of the n dice can roll into 6 possible "types"

1	2	3	4	5	6
oo	o		oooo		ooo

hence # of possible outcomes is:

$$\binom{n+(6-1)}{6-1} = \binom{n+5}{5}$$

So if we roll 10 dice # is:

$$\binom{15}{5} = 3003 \checkmark$$

Ex: How many anagrams of the word

LIMITING

are there?

Sol'n: Two approaches: (1) first distinguish the I's w/ subscripts I_1, I_2, I_3

- # of anagrams w/ distinguished I's is just $8!$

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- For each anagram w/ distinguished I's there are $3!$ equivalent anagrams w/ I's not distinguished

(one for each permutation of I_1, I_2, I_3)

$$\Rightarrow \# \text{ of anagrams} = \frac{8!}{3!}$$

② Can think of anagram being formed in two stages:

- (i) pick 3 positions for the I's $\binom{8}{3}$
- (ii) for remaining 5 positions pick an ordering of LMTNG: $5!$

$$\begin{aligned} \Rightarrow \# \text{ of anagrams} &= \binom{8}{3} \cdot 5! \\ &= \frac{8!}{3!5!} = \frac{8!}{3!} \\ &= 6720, \text{ as before.} \end{aligned}$$

Counting in Two Ways:

Thm (Pascal's identity) Fix $n, k \in \mathbb{N}$
w/ $k \leq n$.

then:

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$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

PF: Let S = the set of k element subsets of $[n] = \{1, 2, \dots, n\}$

$$\text{Then } |S| = \binom{n}{k}$$

OTOH we can partition S into S_1 and T where:

S_1 = k element subsets of $[n]$ that contain 1

T = k element subsets of $[n]$ that do not contain 1.

$$\text{Then } |S| = |S_1| + |T|$$

- subsets in S_1 are formed by selecting $k-1$ elements from $\{2, 3, \dots, n\}$ (1 is automatically included)

$$\Rightarrow |S_1| = \binom{n-1}{k-1}$$

- subsets in T are formed by selecting k elements from $\{2, \dots, n\}$

$$\Rightarrow |T| = \binom{n-1}{k}$$

Hence $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$

i.e. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \checkmark$