Combinatorics

→ the study of counting finite sets
→ easy right? No.

Notation: if A is a finite set, $|A|$ denotes # of el'ts in A

e.g. $|\{*, \heartsuit, \Delta\}| = 3$.

Def'n: A **partition** of a finite set A is a collection of pairwise disjoint subsets $\{A_1, A_2, \ldots, A_k\}$ s.t. $\bigcup_{i=1}^{k} A_i = A$

→ we now allow some pieces $A_i = \emptyset$, o/w same def'n as before.

e.g. if $A_1 = \{*, \Delta\}$ $A_2 = \emptyset$ $A_3 = \{\heartsuit\}$
then $\{A_1, A_2, A_3\}$ is a partition of $A = \{*, \heartsuit, \Delta\}$

Principle (Rule of sum) if $\{A_1, A_2, \ldots, A_k\}$ is a partition of a finite set A, then

$$|A| = \sum_{i=1}^{k} |A_i| = |A_1| + |A_2| + \cdots + |A_k|$$

Pf: obvious

Principle (Rule of product): if the elements of a finite set A are formed by making a sequence of k choices s.t.

① The ith choice can be made in r_i-many ways.

② each el't in A is uniquely formed by such a sequence of choices

Then: $|A| = r_1 r_2 \cdots r_k$

$$= \prod_{i=1}^{k} r_i$$

Pf: not illuminating

Ex ① How many strings of 4 letters (4 letter "words") can be formed using the English alphabet? (e.g. ZZAP and EEEE)

Sol'n: by rule of product, # of such words is $26 \cdot 26 \cdot 26 \cdot 26$

$$= 456{,}976$$

2. How many strings of 4 or fewer letters can be formed?

Sol'n Let A be the set of such strings.

We can partition A as:

$$A = A_1 \cup A_2 \cup A_3 \cup A_4$$

where A_i = set of strings of length i.

By the rule of sum:

$$|A| = |A_1| + |A_2| + |A_3| + |A_4|$$

By rule of product:

$|A_1| = 26$ $\quad |A_2| = 26 \cdot 26$ $\quad |A_3| = 26^3$ $\quad |A_4| = 26^4$

so: $|A| = 26 + 26^2 + 26^3 + 26^4$

$= 475,254$.

Permutations and arrangements

Def'n: if A is a finite set, a permutation of A, is an ordered list of el'ts of A s.t. every el't appears exactly once.

e.g. if $A = \{1, 2, 3\}$ then 213 and 321 are permutations of A; 23 and 2231 are not.

<u>Prop'n</u>: Fix $n \in \mathbb{N} \cup \{0\}$. If A has size n, then the # of permutations of A is $n!$

<u>Pf</u>: - if $|A| = 0$, then $A = \emptyset$ and there is only $1 = 0!$ permutation of A (the empty permutation).

- if $|A| = n \geq 1$, then a permutation

$$a_1 a_2 \cdots a_n$$

is formed by making n choices:

$$\text{choose } a_1, \quad \text{choose } a_2, \quad \ldots, \quad \text{choose } a_n$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$n \text{ choices} \qquad n-1 \text{ choices} \qquad 1 \text{ choice}$$

- so by ROP: # of perms is

$$n(n-1)(n-2)\cdots 1 = n! \checkmark$$

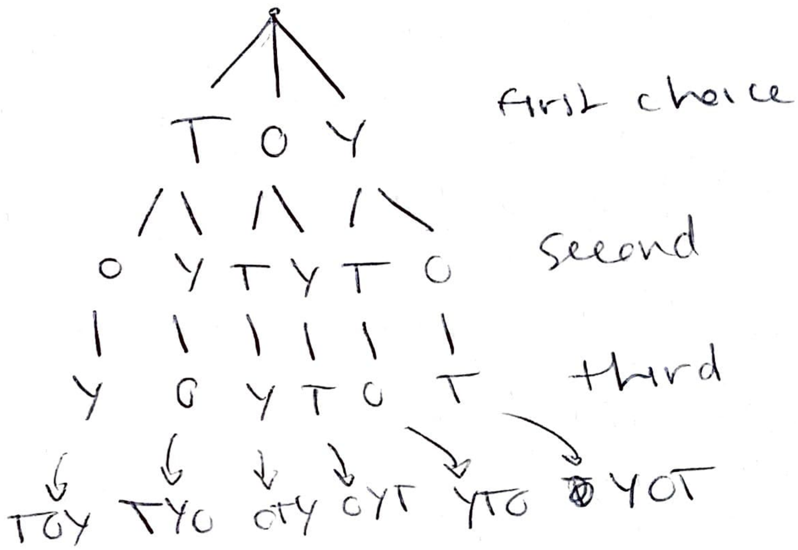
<u>Ex</u>: How many anagrams of the word TOY are there?

<u>Sol'n</u>: an anagram is just a permutation of $\{T, O, Y\}$ (important: TOY has no repeated letters!)

By prop'n, # of anagrams is $3! = 6$

We can list them:

TOY   OTY   YTO
TYO   OYT   YOT



Def'n. Fix $k, n \in \mathbb{N} \cup \{0\}$ with $k \leq n$.
If A has size n, a k-arrangement of A is an ordered list of k elements of A w/ no repeats.

e.g. if $A = \{1,2,3,4,5,6\}$

- then 254 and 152 are 3-arrangements of A
- 115 and 29 are not arrangements of A

Prop'n The # of k-arrangements of a set of size n is $\frac{n!}{(n-k)!} = n \cdot (n-1) \cdots (n-(k-1))$

Pf: such a k-arrangement

$a_1 a_2 \cdots a_k$

formed a sequence of k choices

choose a_1 choose a_2 ... choose a_k

n choices $n-1$ $n-(k-1)$ choices

By RoP # of such arrangements is:

$$n(n-1)\cdots(n-(k-1)) = \prod_{i=0}^{k-1}(n-i)$$

$$= n!/(n-k)! \checkmark$$

Ex: How many 3-letter strings can be formed w/ English alphabet, if no letters repeated?

Sol'n such a string is just a 3-arrangement of $\{a, b, \ldots, z\}$

So: # of such strings is

$$26 \cdot 25 \cdot 24 = 15{,}600.$$

Selections

Def'n: Fix $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$. Given a set A of size n, a k-selection (or k-combination) of A is a subset of A of size k (i.e. an unordered list of el'ts of A)

e.g. if $A = \{1,2,3,4\}$ then $\{2,3\}$ is a 2-selection of A.

Notation $\binom{n}{k}$ denotes the # of k-selections from a set of size n.

Prop'n Given $k \leq n$ we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Pf: - Let A be a set of size n, and let S be the set of k-arrangements of A.

- We will count $|S|$ in two ways

- from prev. prop'n we knew

$$|S| = \frac{n!}{(n-k)!}$$

OTOH: an element of S (i.e. a k-arrangement of A) can be formed in two steps:

first choose a k-selection $\{a_1, a_2, \dots, a_k\}$ $\subseteq A$.

then choose an ordering (i.e. permutation) of this selection.

- there are (by def'n) $\binom{n}{k}$ many ways to make first choice, and $k!$-many ways to make the second.

- Hence, by RoP:

$$|S| = k! \binom{n}{k}$$

$$\Rightarrow \frac{n!}{(n-k)!} = k! \binom{n}{k}$$

$$\Rightarrow \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

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e.g. $\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}\, 3 \cdot 2 \cdot 1}$

$= 120$

notice $\binom{10}{7} = \frac{10!}{3!\,7!} = \binom{10}{3}$ In general: $\binom{n}{k} = \binom{n}{n-k}$

Ex: How many ways are there to choose a chief and two hench persons from a group of 10 ppl?

Sol'n: - 10 choices for chief

- after chief chosen, $\binom{9}{2}$ choices for henchppl.

$\Rightarrow$ # of such committees is

$$10 \cdot \binom{9}{2} = 10 \times \frac{9!}{7!\,2!}$$

$$= 10\left(\frac{9 \cdot 8}{2 \cdot 1}\right)$$

$$= 360$$

Alt Sol'n: - first select group of 3, then from these select chief.

$\Rightarrow$ # of such committees is:

$$\binom{10}{3}\binom{3}{1} = 120 \cdot 3 = 360!$$

as before!