# Finiteness + Infiniteness

**Def'n** • A set $X$ is finite iff

① $X = \emptyset$ or

② $\exists n \in \mathbb{N}$ and a bijection $f : [n] \to X$.

• $X$ is infinite iff it is not finite, i.e.

iff

① $X \neq \emptyset$ and

② $\forall n \in \mathbb{N}$ there is no bijection $f : [n] \to X$.

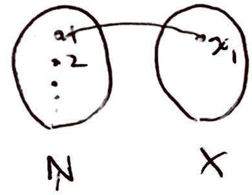
The following theorem says $\mathbb{N}$ has the "smallest" possible infinite size.

**Thm** if $X$ is infinite then $\mathbb{N} \preceq X$.

**Pf**: we define an injection $f : \mathbb{N} \to X$ inductively

(BC) Since $X \neq \emptyset$, there is $x_1 \in X$.

let $f_1 = \{(1, x_1)\}$



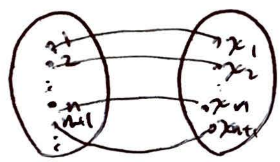

(IH) Sps at stage $n$, we have defined a "partial injection" $f_n$, i.e.

$f_n = \{(1, x_1), (2, x_2), \ldots, (n, x_n)\}$ s.t. if $i \neq j$ then $x_i \neq x_j$

(IS) We can view $f_n$ as an injection from $[n]$ to $X$.
Since $X$ is not finite, it must be that $f_n$ is **not** a surjection (o/w it would be a bijection)

i.e. $\exists x_{n+1} \in X$ s.t. $x_{n+1} \notin \{x_1, \dots, x_{n+1}\}$

So define $f_{n+1} = \{(1,x_1), (2,x_2), \dots, (n,x_n), (n+1,x_{n+1})\}$



$\mathbb{N}$

Continuing inductively we can define a sequence of injections $f_n : [n] \to X$ s.t. $\forall n \leq m$, $f_n \subseteq f_m$.

Now let $f = \bigcup f_n = \{(1,x_1), (2,x_2), \dots\}$

then $f : \mathbb{N} \to X$ is an injection ✓

<u>Note</u>! This proof is a bit informal; we're not only using induction, but also what set theorists call the <u>recursion theorem</u> as well as the axiom of choice (AC).

- Theorem says: $\mathbb{N}$ is as small as possible for an infinite set: even sets that may appear smaller (e.g. $E$, $O$, or other strict subsets of $\mathbb{N}$) are actually not.
- OTOH, many sets which appear larger than $\mathbb{N}$ (e.g. $\mathbb{Z}$, $\mathbb{Q}$, ...) are actually the same size.

__Def'n__ A set $X$ is called __countable__ iff $\mathbb{N} \sim X$.

__Ex's__ ① $\mathbb{Z}$ is countable.

__Pf__: we showed

$$f : \mathbb{Z} \to \mathbb{N} \qquad f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$

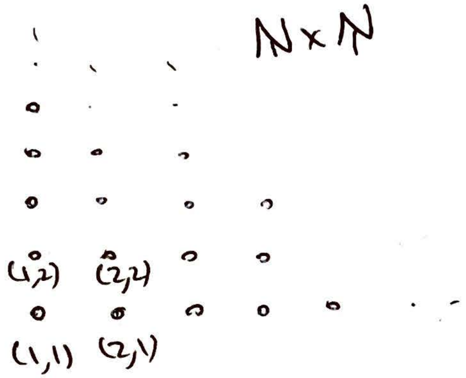
is a bijection.

② $\mathbb{N} \times \mathbb{N}$ is countable

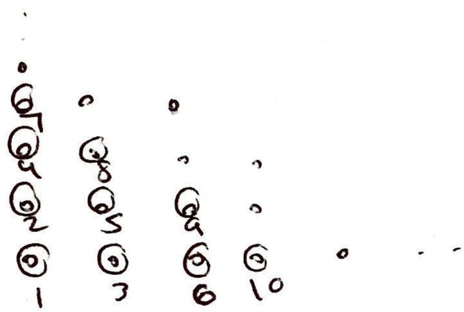
__Pf__: need to construct a bijection

$f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

↪ possible to do this explicitly (i.e. define $f$ w/ a formula), but we just draw a picture.




to construct $f: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ we "count $\mathbb{N}\times\mathbb{N}$ across its diagonals"




i.e. $f(1) = (1,1)$
$f(2) = (1,2)$
$f(3) = (2,1)$
$f(4) = (1,3)$ etc...

resulting $f: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is injective/surjective

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↳ sometimes it's hard to show $A \sim B$ directly (i.e. build a bijection) but easy to show $A \lesssim B$ and $B \lesssim A$.

↳ turns out: this is good enough to guarantee $A \sim B$!

<u>Theorem</u> (Cantor-Schroeder-Bernstein)
For any sets $A, B$, if $A \lesssim B$ and $B \lesssim A$ then $A \sim B$.

Pf: interesting but tricky – we'll put it off for now and take theorem for granted.

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CSB says: $A \lesssim B \wedge B \lesssim A \Rightarrow A \sim B$

Since we know: $A \lesssim B$ iff $B \gtrsim A$, CSB also gives: $B \gtrsim A \wedge A \gtrsim B \Rightarrow A \sim B$.

i.e. CSB says: $\lesssim$ and $\gtrsim$ are "antisymmetric up to $\sim$"

Next goal: prove that $\mathbb{N} \sim \mathbb{Q}$!

First need:

Theorem: If $A, B$ are ctbl sets, then $A \times B$ is ctbl.

Pf: Sps $A, B$ are ctbl, i.e. we have bijections

$$f: \mathbb{N} \to A$$
$$g: \mathbb{N} \to B$$

We know: $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$

So if we can show: $\mathbb{N} \times \mathbb{N} \sim A \times B$, we'll be done.

Consider: $F: \mathbb{N} \times \mathbb{N} \to A \times B$ defined by:

$$F(n, m) = (f(n), g(m))$$

Claim $F$ is a bijection

Pf: (surj) - Fix $(a,b) \in A \times B$
- since $f, g$ both surj. $\exists\, n \in \mathbb{N}$, $m \in \mathbb{N}$
s.t. $f(n) = a$ $g(m) = b$
- hence $F(n,m) = (a,b)$ ✓

(inj) - if $F(n,m) = F(n',m')$
- then $(f(n), g(m)) = (f(n'), g(m'))$
i.e. $f(n) = f(n')$
and $g(m) = g(m')$
hence $n = n'$ and $m = m'$ ($f, g$ both inj.)
i.e. $(n,m) = (n',m')$ ✓

Hence $\mathbb{N} \times \mathbb{N} \sim A \times B$
hence $\mathbb{N} \sim A \times B$, as desired.

Theorem $\mathbb{Q}$ is ctbl, i.e. $\mathbb{N} \sim \mathbb{Q}$.

Pf: We'll prove:
$$\mathbb{N} \overset{①}{\lesssim} \mathbb{Q} \overset{②}{\lesssim} \mathbb{Z} \times \mathbb{N} \overset{③}{\lesssim} \mathbb{N}$$
By transitivity this gives $\mathbb{N} \lesssim \mathbb{Q} \lesssim \mathbb{N}$
which gives $\mathbb{N} \sim \mathbb{Q}$ by CSB.

① holds since $\mathbb{Q}$ is infinite, by a previous theorem

③ holds by immediately previous theorem: in fact we know $\mathbb{Z}\times\mathbb{N}\sim\mathbb{N}$, since $\mathbb{Z}\sim\mathbb{N}$ and $\mathbb{N}\sim\mathbb{N}$.

So remains to prove ②. We prove $\gtrsim$ version.

<u>Claim</u>: $\mathbb{Z}\times\mathbb{N}\gtrsim\mathbb{Q}$.

Pf: $F:\mathbb{Z}\times\mathbb{N}\to\mathbb{Q}$ defined by $F(m,n)=\frac{m}{n}$ is a surjection.

<u>Why</u>: if $q\in\mathbb{Q}$ and $q=\frac{m}{n}$ then $F(m,n)=q$!

→ F is <u>not</u> injective, e.g. $F(1,2)=F(3,4)=F(3,6)=\ldots$
→ who cares! we've still shown $\mathbb{Z}\times\mathbb{N}\gtrsim\mathbb{Q}$

→ by above, we're done proving $\mathbb{Q}\sim\mathbb{N}$!

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<u>Summary</u>: - among infinite sets X, $\mathbb{N}$ is "small" in the sense that $\mathbb{N}\lesssim X$ always

- OTOH $\mathbb{N}$ is "large": many sets X which seem larger than $\mathbb{N}$ (e.g. $\mathbb{Z}$, $\mathbb{Q}$) we actually have $X\sim\mathbb{N}$.

Q: Are there infinite sets X s.t. $\mathbb{N}\not\sim X$?

A: yes!

<u>Theorem</u> (Cantor): $\mathbb{N} < P(\mathbb{N})$. That is:

$\mathbb{N} \lesssim P(\mathbb{N})$ but $\mathbb{N} \not\approx P(\mathbb{N})$.

<u>Pf</u>: we knew $\mathbb{N} \lesssim P(\mathbb{N})$ since $P(\mathbb{N})$ is infinite

So need to show $\mathbb{N} \not\approx P(\mathbb{N})$

<u>We prove</u>: <u>Claim</u> Let $f: \mathbb{N} \to P(\mathbb{N})$ be a function (<u>any</u> function). Then: $f$ is <u>not</u> a surjection.

Pf: a magic trick:

Let $T = \{n \in \mathbb{N} \mid n \notin f(n)\}$

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<u>to illustrate</u>: e.g. if

$f(1) = \{1, 7, 10\}$
$f(2) = \{1, 3, 5, 7, \ldots\}$
$f(3) = \emptyset$
$f(4) = \{2, 4, 6, 8, \ldots\}$

<u>then</u>: $1 \notin T$ because $1 \in f(1)$
$2 \in T$ since $2 \notin f(2)$
$3 \in T$ since $3 \notin f(3)$
$4 \notin T$ since $4 \in f(4)$

So $T = \{2, 3, \ldots\}$ in this case

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<u>Then</u>: $(\forall n \in \mathbb{N})\ f(n) \neq T$

<u>Pf</u>: Fix $n \in \mathbb{N}$. <u>Case 1</u>: if $n \in T$ then $n \notin f(n)$ by def'n of $T$. Hence $T \neq f(n)$ in this case $\left(\begin{smallmatrix} n \in T \\ n \notin f(n) \end{smallmatrix}\right)$

<u>Case 2</u>: if $n \notin T$ then $n \in f(n)$, again by def'n of $T$. Hence $T \neq f(n)$ again

$\hookrightarrow$ since $n$ was arbitrary we have $f(n) \neq T$ for every $n$.

But now the claim follows: $T \notin \mathrm{Im} f$, hence $f$ is not surjective. ✓

Since $f$ was arbitrary there is <u>no</u> surjection $f: \mathbb{N} \to P(\mathbb{N})$ (hence no bijection)

<u>hence $\mathbb{N} \not\approx P(\mathbb{N})$</u>. ✓

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Same proof works in general:

<u>Theorem</u>: For any set $A$, there is no surjection $f: A \to P(A)$.

<u>Pf</u>: Fix $f: A \to P(A)$

Let $T = \{a \in A \mid a \notin f(a)\}$

Then (same arg): $\forall a \in A$, $f(a) \neq T$. ✓

Since $g: A \to P(A)$, $g(a) = \{a\}$ always defines an injection, above theorem shows

$$A < P(A) \quad \text{for every set } A.$$

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It follows there are infinitely many levels of infinity:

$$\mathbb{N} < P(\mathbb{N}) < P(P(\mathbb{N})) < \cdots$$

<u>Def'n</u> if $X$ is infinite and $\mathbb{N} \not\approx X$, we say $X$ is <u>uncountable</u>.

Above: $P(\mathbb{N})$ is unctbl.

Other examples?

<u>Sets</u> <u>of functions</u>:

Consider the set $F$ of functions $f: \mathbb{N} \to \{0,1\}$

$$F = \{f \subseteq \mathbb{N} \times \{0,1\} \mid f \text{ is a function}\}$$

- we can think of a given $f \in F$ as an infinite 01-sequence

e.g. if
$f(1) = 0$
$f(2) = 0$
$f(3) = 1$
$f(4) = 0$
$f(5) = 1$
$\vdots$

can picture $f$ like this:

$f = 00101\ldots$

and so if I write:

$$g = 101010\ldots$$

I mean $g$ is the function s.t.

$$g(1) = 1$$
$$g(2) = 0$$
$$g(3) = 1$$
$$g(4) = 0 \quad \text{etc.}$$
$$\vdots$$

<u>Theorem</u> $F$ is unctbl.

Pf: Diagonalize!

<u>Claim</u>: if $H: \mathbb{N} \to F$ is a function, then $H$ is <u>not</u> a surjection

Pf: define a function $f \in F$ as follows:

$$f(n) = \begin{cases} 1 & \text{iff } H(n)(n) = 0 \\ 0 & \text{iff } H(n)(n) = 1 \end{cases}$$

then (by def'n) $(\forall n \in \mathbb{N})\ f(n) \neq H(n)(n)$

hence: $(\forall n \in \mathbb{N})\ f \neq H(n)$!

$f$, $H(n)$ differ in the nth place

the claim follows. ✓

e.g. if

$H(1)(1)$ $\quad H(2)(2)$ $\quad H(3)(3)$ $\quad H(4)(4)$

$$\begin{aligned}
H(1) &= 0 \quad 1 \quad 0 \quad 1 \quad \ldots \\
H(2) &= 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad \ldots \\
H(3) &= 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad \ldots \\
H(4) &= 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad \ldots \\
&\vdots
\end{aligned}$$

then would have $f = 1100\ldots$ in this case.