- bijections
- requiz, HW

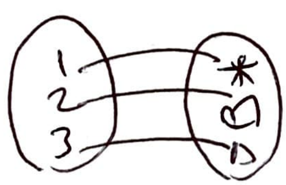
Infinity

The concept of cardinality:

- we would say the set $\{*, \heartsuit, \Delta\}$ has 3 elements, or is of size 3.
- why? By counting it!

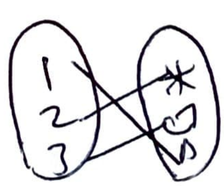
$$\{*, \heartsuit, \Delta\}$$
$$1 \quad 2 \quad 3$$

- in doing so, we are implicitly defining a bijection between the sets $\{1,2,3\}$ and $\{*, \heartsuit, \Delta\}$



- we could've counted differently:

$$\{*, \heartsuit, \Delta\}$$
$$2 \quad 3 \quad 1$$



- generalizing this idea: we'll say two sets have the same size iff there is a bijection between them

Def'n: We say that two sets A, B have the same cardinality, and write $A \sim B$, iff there is a bijection $f: A \to B$.

Note: - In set theory courses, one defines, for every set A, the cardinal number $|A|$.
- can then prove: $A \sim B$ iff $|A| = |B|$.
(e.g. $|\{*, \triangleleft, *\}| = |\{\circ, \square, \diamond\}| = 3$)
- defining cardinal #s beyond our scope; for us: $|A| = |B|$ just means $A \sim B$, i.e. $\exists f: A \to B$ a bijection.

Properties of $\sim$: ① For any set A, $id_A: A \to A$ is a bijection (why?)
- Hence $A \sim A$, i.e. $\sim$ is reflexive.

② If $f: A \to B$ is a bijection, then f is invertible and $f^{-1}: B \to A$ is a bijection.
- Hence: if $A \sim B$, then $B \sim A$, i.e. $\sim$ is symmetric

③ On HW you showed: if $f: A \to B$ and $g: B \to C$ are bijections then $g \circ f: A \to C$ is a bijection

- Hence: if $A \sim B$ and $B \sim C$, then $A \sim C$, i.e. $\sim$ is <u>transitive</u>

→ ①+②+③ say: $\sim$ is an equivalence relation on sets!

→ $\sim$ is most interesting when the sets being compared are <u>infinite</u>.

<u>Ex's</u>: ① $\{1,2,3\} \sim \{*, \heartsuit, \Delta\}$ since $f: \{(1,*), (2,\heartsuit), (3,\Delta)\}$ is a bijection.

② Let $-\mathbb{N}$ denote the set $\{-1,-2,-3,\ldots\}$

define: $f: \mathbb{N} \to -\mathbb{N}$

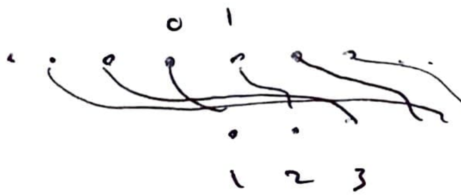
$$f(n) = -n$$

→ ez to check: f is a bijection.

Hence $\mathbb{N} \sim -\mathbb{N}$.

③ Last time we proved: $f: \mathbb{Z} \to \mathbb{N}$ defined by

$$f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$



is a bijection. Hence $\mathbb{Z} \sim \mathbb{N}$.

④ Combining ② and ③ gives: $\mathbb{Z} \sim -\mathbb{N}$.
by transitivity of $\sim$.

<u>Def'n</u>: Let A, B be sets.

① We write $A \lesssim B$ (or $|A| \leq |B|$) to mean: there is an <u>injection</u> $f: A \to B$

② We write $A \gtrsim B$ (or $|A| \geq |B|$) to mean: there is a <u>surjection</u> $g: A \to B$.

↳ we'll write $A < B$ to mean: there is an injection $f: A \to B$ but no bijection $g: A \to B$ (i.e. $A \lesssim B$ but $A \not\sim B$)

↳ similarly for $A > B$.

$\boxed{\text{NOTE!}}$ $A \gtrsim B$ is <u>not</u> the "reverse of" $A \lesssim B$,
i.e. it is <u>not</u> asserting there is an injection from B to A.

<u>But this follows</u>:

<u>Theorem</u>: For all sets A, B we have:

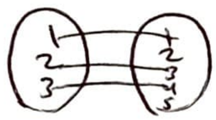
$$A \lesssim B \text{ iff } B \gtrsim A$$

i.e. there is an injection $f: A \to B$ iff there is a surjection $g: B \to A$.

Before proving theorem, let's do some examples to illustrate ideas:

<u>ex</u>: (1) Consider $f : \underbrace{\{1,2,3\}}_{=[3]} \to \underbrace{\{1,2,3,4,5\}}_{=[5]}$

defined by: $f(1) = 1$, $f(2) = 3$, $f(3) = 4$



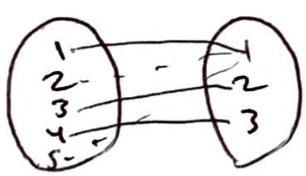
<u>Observe</u>: f is an injection. Hence $[3] \lesssim [5]$.

<u>idea</u>: to get a surjection $g : [5] \to [3]$

we "reverse" f then map anything left over to something arbitrary: e.g.

let
$$\left.\begin{aligned} g(1) &= 1 \\ g(3) &= 2 \\ g(4) &= 3 \end{aligned}\right\} \text{"reverse" of } f$$

$$\left.\begin{aligned} g(2) &= 1 \\ g(5) &= 1 \end{aligned}\right\} \text{could put anything here}$$



then: g is a <u>function</u> since f was injective, and is clearly surjective ✓

can use this idea to prove more generally

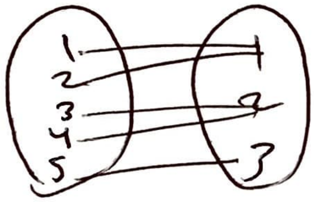
if $A \lesssim B$ then $B \gtrsim A$.

② Consider: $g:[5]\to[3]$ defined by:

$g(1) = g(2) = 1$

$g(2) = g(3) = 2$

$g(5) = 3.$

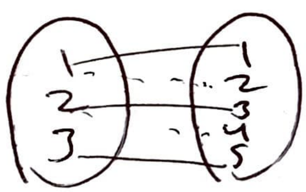


observe: g is a surjection hence $[5] \gtrsim [3]$.

idea: to get an injection $f:[3]\to[5]$ we ~~take~~ some "reverse" of g w/o repeats.

e.g. define: $f:[3]\to[5]$ by

$f(1) = 1$, $f(2) = 3$, $f(3) = 5$



"reverses" g w/o repeats

then: f is a function because g was surjective, and because we deleted repeats. (It's injective because g was a function)

Same idea can be used to prove

if $B \gtrsim A$ then $A \lesssim B$.

Let's now give a formal proof of the theorem:

<u>Pf</u> ($\Rightarrow$) Spse $A \lesssim B$, i.e. $\exists f : A \to B$ an injection. We WTS $\exists g : B \to A$ a surjection i.e. $B \gtrsim A$.

Define g as follows:

- first, fix some $a_0 \in A$
- Consider a fixed $b \in B$.

If $b \in \mathrm{Im}_f$, then $\exists a \in A$ s.t. $f(a) = b$. Moreover this a is <u>unique</u>, since f is injective.

In this case, define $g(b) = a$.

If $b \notin \mathrm{Im} f$, define $g(b) = a_0$

Then: ① g is a function from B to A

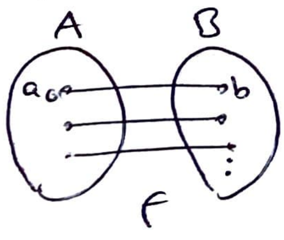
(if $b \in \mathrm{Im} f$ then $g(b)$ = unique $a \in A$ s.t. $f(a) = b$

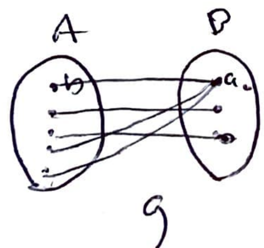
if $b \notin \mathrm{Im} f$ then $g(b) = a_0$)

② g is surjective

(Fix $a \in A$. Let $b = f(a)$. then by def'n $g(b) = a$ ✓)

Picture:





($\Leftarrow$) Now suppose $B \gtrsim A$, i.e. there is a surjection $g: B \to A$. We WTS there is an injection $f: A \to B$.

Define f as follows:

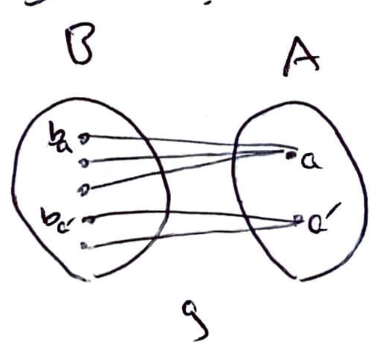
- for a given $a \in A$, since g is surjective there is at least one $b \in B$ s.t. $g(b) = a$, i.e. $\text{PreImg}_g(\{a\}) \neq \emptyset$.
- so pick one distinguished element $b_a \in \text{PreImg}_f(\{a\})$ and define $f(a) = b_a$. (observe $g(b_a) = a$)
- do this for every $a \in A$.

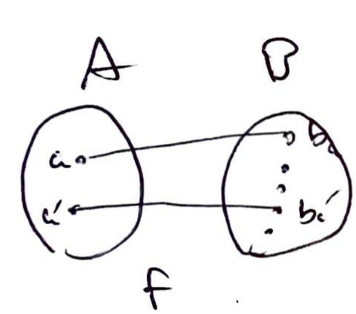
<u>Then</u>: ① f is a function from A to B: every $a \in A$ has a unique output $b_a \in B$

from $\text{PreImg}_g(\{a\})$

② f is injective: if $f(a) = f(a') = b$ then $g(b) = a$, $g(b) = a'$ hence $a = a'$ ✓ since g is a function.

<u>Picture</u>





Properties of $\lesssim$ and $\gtrsim$

Suppose A, B, C are sets.

1. $A \lesssim A$ and $A \gtrsim A$ since $id_A : A \to A$ is both an injection and surjection.
Hence $\lesssim$ and $\gtrsim$ are reflexive.

2. If $A \lesssim B$ and $B \lesssim C$ then $\exists f : A \to B$ and $\exists g : B \to C$ injection. By HW, $g \circ f : A \to C$ is an injection. Hence $A \lesssim C$.

Sim'ly if $A \gtrsim B \gtrsim C$ then $A \gtrsim C$.

Hence $\lesssim$ and $\gtrsim$ are transitive.

3. Are they antisymmetric? Not literally.

If $A \lesssim B$ and $B \lesssim A$ not necessarily that $A = B$.

e.g. $A = \{1, 2, 3\}$ $B = \{*, ♡, \Delta\}$.

$\boxed{\text{But!!}}$ We'll show in this case that $A \sim B$ always.

4. Are they total? That is, for any two sets A, B do we always have $A \lesssim B$ or $B \lesssim A$?

Yes! If you assume axiom of choice (but this beyond scope)

Notation: $|A| = |B|$ means $A \sim B$

may also write:

$|A| \leq |B|$ for $A \precsim B$

$|A| \geq |B|$ for $A \succsim B$.

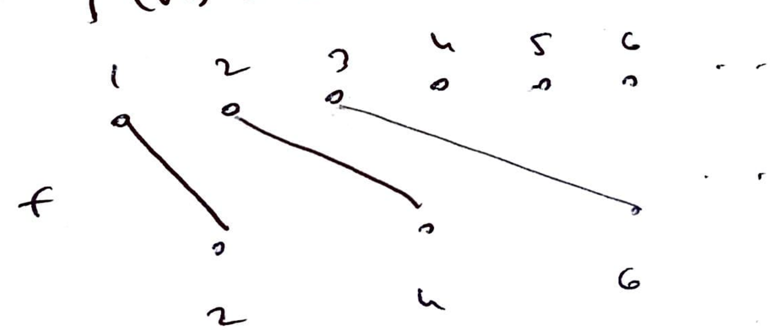
Some paradoxes of infinity

Theme: $A \sim B$ when A, B are infinite can be counterintuitive!

① Let $E = \{2, 4, 6, \ldots\}$

Then $\mathbb{N} \sim E$ ("there are as many even numbers as whole numbers")

Why: $f : \mathbb{N} \to E$ defined by $f(n) = 2n$ is a bijection:



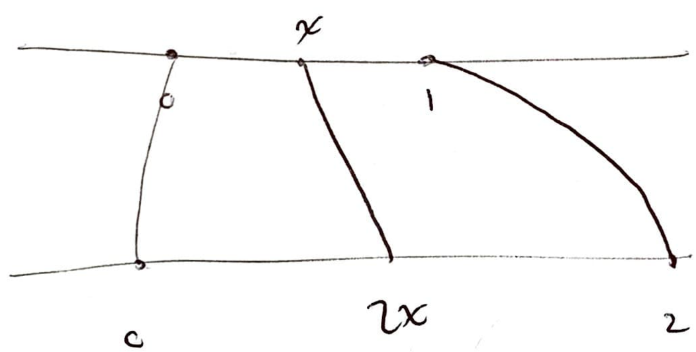
② Let $S = \{1, 4, 9, \ldots\}$ = set of squares

then $\mathbb{N} \sim S$ ("there are as many squares as whole numbers")

<u>Why</u>: $f: \mathbb{N} \to S$ defined by $f(n) = n^2$ is a bijection.

③ We have $[0,1] \sim [0,2]$ ("There are as many numbers between 0 and 1 as between 0 and 2")

Why: $f: [0,1] \to [0,2]$ defined by $f(x) = 2x$ is a bijection.

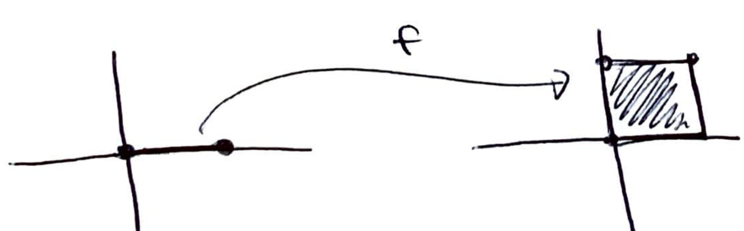


④ In fact: "The side is as large as the square" i.e. $[0,1] \sim [0,1] \times [0,1]$

i.e. there is a bijection

$f: [0,1] \to [0,1] \times [0,1]$

<u>Why</u>: beyond our scope



moral: infinite sets are wild!