hence $f(n) \neq f(m)$ in this case as well.

↳ hence in all cases $f(n) \neq f(m)$

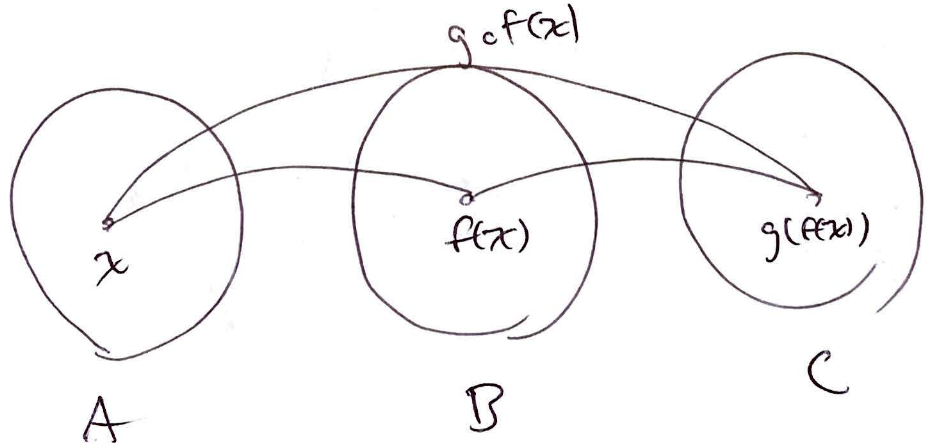
↳ since n, m were arbitrary, we've proved f is injective. ✓

hence f is bijective ✓

Compositions

Def'n: Sps $f: A \to B$ and $g: B \to C$ are functions. The composition of f and g, denoted $g \circ f$, is defined by; $\forall x \in A$

$$g \circ f(x) = g(f(x))$$



Observe, so defined we have that $g \circ f$ is a function from A to C.

(xxiv)

<u>ex</u>: Define: $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by

$$f(m,n) = m+n$$

$g: \mathbb{Z} \to \mathbb{N}$ by

$$g(n) = n^2+1$$

<u>then</u>: $g \circ f(1,3) = g(f(1,3))$

$= g(1+3)$

$= g(4) = 4^2+1 = 17$

and in <u>general</u>: $g \circ f(m,n) = g(f(m,n))$

$= g(m+n)$

$= (m+n)^2+1.$

and (observe!) $g \circ f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$

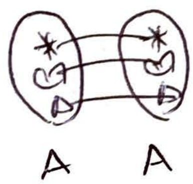
(xxv) The identity function:

Def'n: Let A be a set. The identity function on A, denoted id_A is defined by:

$$id_A : A \to A$$
$$id_A(x) = x$$

e.g. if $A = \{*, ♡, \Delta\}$ then $id_A : A \to A$ is:

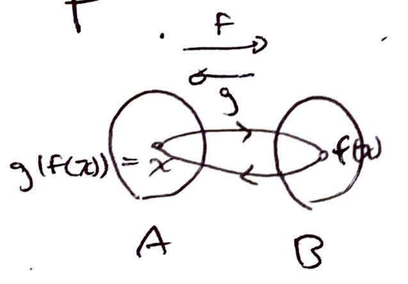
$$id_A = \{(*,*), (♡,♡), (\Delta,\Delta)\}$$



Def'n: a function $f : A \to B$ is called invertible iff there is a function $g : B \to A$ s.t.:

$$g \circ f = id_A \quad \text{i.e. } \forall x \in A \quad g(f(x)) = x$$
$$f \circ g = id_B \quad \text{i.e. } \forall y \in B \quad f(g(y)) = y.$$

g is called the inverse of f and is denoted f^{-1}.



(xxxvi)

Ex: Consider $f: \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = 2x+1.$$

Let $g: \mathbb{R} \to \mathbb{R}$ be defined by
$$g(x) = \frac{x-1}{2}.$$

Observe: $\forall x \in \mathbb{R}$
$$g(f(x)) = g(2x+1) = \frac{(2x+1)-1}{2} = x \Rightarrow \boxed{\text{so } g \circ f = id_{\mathbb{R}}}.$$

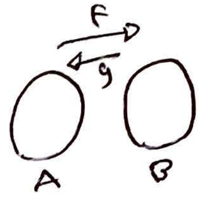
and $\forall x \in \mathbb{R}$
$$f(g(x)) = f\left(\frac{x-1}{2}\right) = 2\left(\frac{x-1}{2}\right)+1 = x \Rightarrow \boxed{\text{so } f \circ g = id_{\mathbb{R}}}$$

this shows: f is invertible and its inverse f^{-1} is g.

Note: not all functions are invertible!

In fact: **Theorem** a function $f: A \to B$ is invertible if and only if f is a bijection

(xxvii) **Pf:** ($\Rightarrow$) Suppose that f is invertible, and let $g = f^{-1}$ be its inverse.



WTS: f is a bijection

(surjectivity): – Fix $y \in B$.

– Let $x = g(y)$

– then: $f(x) = f(g(y))$

$= y$.

→ since y was arbitrary, f is surjective ✓

(injectivity): – Fix $x, y \in A$ and sps $f(x) = f(y)$

– then: $g(f(x)) = g(f(y))$

– i.e. $x = y$ (since $g \circ f = id_A$)

→ since x, y were arbitrary, f is injective

($\Leftarrow$) Sps $f: A \to B$ is a bijection.

WTS: f is invertible.

Define: $g = \{(b,a) \in B \times A \mid (a,b) \in f\}$

Claim: $g = f^{-1}$.

e.g. if
$f = \{(1,*), (2,♡), (3,\Delta)\}$

$g = \{(*,1), (♡,2), (\Delta,3)\}$

(xxviii)

PF: we first prove: subclaim: g is a function from B to A.

Pf: WTS $\forall b \in B$ $\exists$ a unique $a \in A$ s.t. $(b,a) \in g$

So fix $b \in B$

(existence): - Since f is surjective $\exists a \in A$ s.t. $f(a) = b$, i.e. $(a,b) \in f$

- hence $(b,a) \in g$, by def'n of g

(uniqueness): - Sps there is an $a' \in A$ s.t. $(b,a') \in g$.

- then $(a',b) \in f$ by def'n of g

- i.e. $f(a') = b$

- but then $f(a') = f(a)$. Since f is injective thus implies $a = a'$ ✓

subclaim is proved ✓

Now we show $g = f^{-1}$. (i.e. $g \circ f = id_A$, $f \circ g = id_B$.)

Pf: - Fix $a \in A$

- Let $b = f(a)$ so that $(a,b) \in f$.

- then $(b,a) \in g$, i.e. $g(b) = a$.

so: $g(f(a)) = g(b) = a$

$\Rightarrow$ Since a was arbitrary, $g \circ f = id_A$. ✓

$(x x_1 x)$

- Now fix $b \in B$. Let $a = g(b)$ so that $(b, a) \in g$.
- hence $(a, b) \in f$ i.e. $f(a) = b$.
- So $f \circ g(b) = f(g(b)) = f(a) = b$.
- Since b was arbitrary $f \circ g = id_B$ ✓

hence $g = f^{-1}$, as claimed ✓

We can sometimes use theorem to prove a given f is a bijection.

ex: $f : \mathbb{R} \to \mathbb{R}$
$f(x) = 2x + 1$ is a bijection.

PF: we checked above: if $g(x) = \frac{x-1}{2}$
then $g = f^{-1}$

hence f is invertible.
by thm, f is a bijection (of $\mathbb{R}$ with itself).