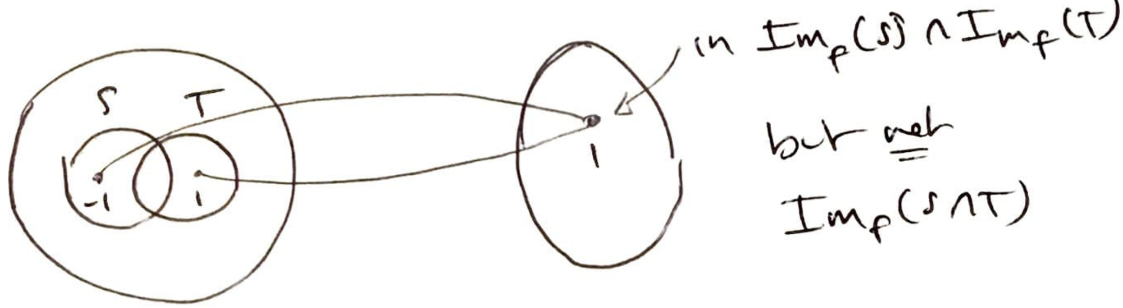
(xi)

Schematically:



PreImages:

Def'n: Sps $f: A \to B$ is a function and $Y \subseteq B$. The preimage of Y, denoted $PreIm_f(Y)$, is defined as:

$$PreIm_f(Y) = \{x \in A \mid f(x) \in Y\}$$

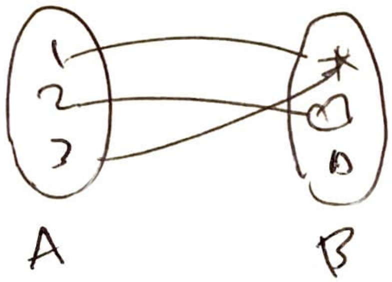
= the inputs in A whose outputs are in Y.

Note: Since $f(x) \in B$ for every $x \in A$, we don't separately define $PreIm_f(B)$ — this is always just A.

ex: ① Let $A = \{1,2,3\}$ $\quad f = \{(1,*), (2,\heartsuit), (3,\emptyset)\}$

$B = \{*, \heartsuit, \emptyset\}$

(xii)



then: $\text{PreImg}_f(\{*\}) = \{x \in A \mid f(x) \in \{*\}\}$

$= \{x \in A \mid f(x) = *\}$

$= \{1, 3\}.$

$\text{PreImg}_f(\{*, \heartsuit\}) = \{x \in A \mid f(x) \in \{*, \heartsuit\}\}$

$= \{1, 2, 3\} = A.$

$\text{PreImg}_f(\{\Delta\}) = \{x \in A \mid f(x) \in \{\Delta\}\}$

$= \{x \in A \mid f(x) = \Delta\}$

$= \phi.$

(2) Consider $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

Then: $\text{PreImg}_f(\{0, 1\}) = \{x \in \mathbb{R} \mid f(x) \in \{0, 1\}\}$

$= \{x \in \mathbb{R} \mid x^2 \in \{0, 1\}\}$

$= \{-1, 0, 1\}.$

Also: $\text{PreImg}_f([0, 2]) = \{x \in \mathbb{R} \mid x^2 \in [0, 2]\}$

$= \{x \in \mathbb{R} \mid 0 \leq x^2 \leq 2\}$

$= \{x \in \mathbb{R} \mid x^2 \leq 2\}$

$= \{x \in \mathbb{R} \mid -\sqrt{2} \leq x \leq \sqrt{2}\}$

(xiii)

$= [-\sqrt{2}, \sqrt{2}]$

Also: $\mathrm{PreIm}_f([0,\infty)) = \{x \in \mathbb{R} \mid x^2 \in [0,\infty)\}$

$= \mathbb{R}$.

Q: What happens if we take the preimage of the image of some $X \subseteq A$?

or the image of the preimage of some $Y \subseteq B$?

Prop'n Sps $f: A \to B$ is a function.

(i) Fix $X \subseteq A$

Then: $\mathrm{PreIm}_f(\mathrm{Im}_f(X)) \supseteq X$

(ii) Fix $Y \subseteq B$

Then: $\mathrm{Im}_f(\mathrm{PreIm}_f(Y)) \subseteq Y$.

Pf: (i) Fix $x \in X$.

By def'n: $\mathrm{PreIm}_f(\mathrm{Im}_f(X))$

$= \{y \in A \mid f(y) \in \mathrm{Im}_f(X)\}$

but since $x \in X$, we know $f(x) \in \mathrm{Im}_f(X)$

by def'n of $\mathrm{Im}_f(X)$

Hence $x \in \mathrm{PreIm}_f(\mathrm{Im}_f(X))$

Since x was arbitrary, (i) is proved. ✓

(xiv) Fix $y \in Im_f(PreIm_f(Y))$

by def'n of image, $\exists x \in PreIm_f(Y)$

s.t. $f(x) = y$.

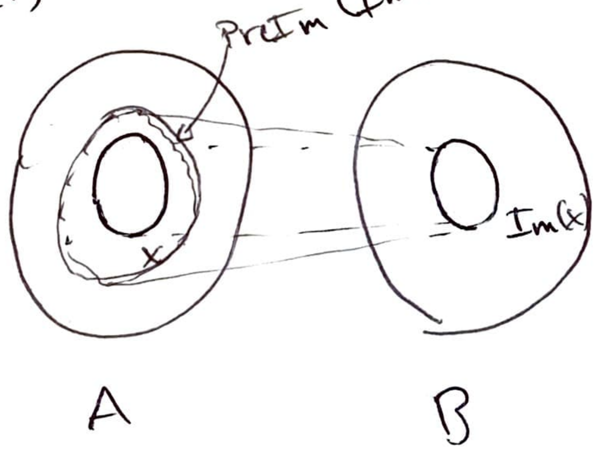
But then, by def'n of preimage,

$f(x) \in Y$, i.e. $y \in Y$.

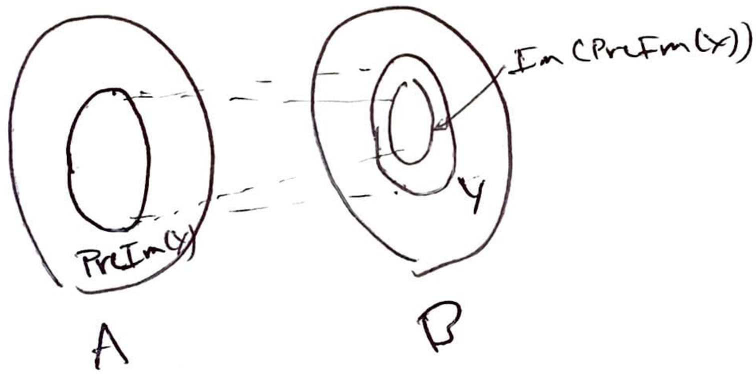
Since y was arbitrary, (ii) is proved ✓

Picture:

(i)



(ii)



Note: in general neither containment can be reversed.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$

be $f(x) = x^2$

Let $X = \{1\}$

(xv)

Then: $\operatorname{Im}_f(X) = \operatorname{Im}_f(\{1\})$

$= \{f(1)\}$

$= \{1^2\} = \{1\}.$

So: $\operatorname{PreIm}_f(\operatorname{Im}_f(X)) = \operatorname{PreIm}_f(\{1\})$

$= \{x \in \mathbb{R} \mid f(x) \in \{1\}\}$

$= \{x \in \mathbb{R} \mid x^2 \in \{1\}\}$

$= \{-1, 1\}.$

hence: $X = \{1\} \subsetneq \{-1, 1\} = \operatorname{PreIm}_f(X)$

in this case. ✓

Now let $Y = \{-2, 1\}$

Then: $\operatorname{PreIm}_f(Y) = \{x \in \mathbb{R} \mid f(x) \in \{-2, 1\}\}$

$= \{x \in \mathbb{R} \mid x^2 \in \{-2, 1\}\}$

$= \{-1, 1\}$

So: $\operatorname{Im}_f(\operatorname{PreIm}_f(Y)) = \operatorname{Im}_f(\{-1, 1\})$

$= \{f(-1), f(1)\}$

$= \{(-1)^2, 1^2\}$

$= \{1\}$

So $\operatorname{Im}_f(\operatorname{PreIm}_f(Y)) = \{1\} \subsetneq \{-2, 1\} = Y$

in this case ✓

(xvi) **—jections**

Let $A = \{1, 2, 3\}$

$B = \{*, ♡\}$

$C = \{1, 2\}$

$D = \{*, ♡, \Delta\}$.

define: $g: A \to B$

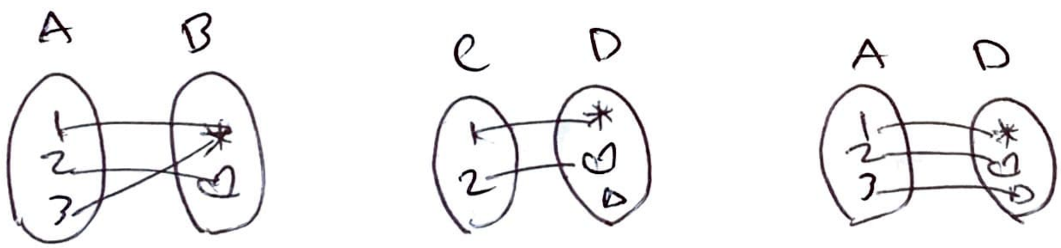
$h: C \to D$

$j: A \to D$

by: $g = \{(1,*), (2,♡), (3,*)\}$

$h = \{(1,*), (2,♡), (3,*)\}$

$j = \{(1,*), (2,♡), (3,\Delta)\}$.



g h j

Surjections: **Def'n**: a function $f: A \to B$ is **surjective** (or **onto**) iff $\operatorname{Im} f = B$.

i.e. iff $(\forall b \in B)(\exists a \in A)(f(a) = b)$.

(xvii) ex's – g and j above are surjective
– h is not; because $\Delta \notin Im_h$

Proving surjectivity

ex: ① Define $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ by $f((m,n)) = m+n$.
write $f(m,n)$

Claim f is surjective

Pf: WTS: $(\forall x \in \mathbb{Z})(\exists (m,n) \in \mathbb{Z}\times\mathbb{Z})(f(m,n) = x)$

- so fix $x \in \mathbb{Z}$
- observe $f(0,x) = 0 + x = x$
- hence $\exists (m,n) \in \mathbb{Z}\times\mathbb{Z}$ s.t. $f(m,n) = x$, namely $(m,n) = (0,x)$.
- since x was arbitrary, claim is proved ✓

② Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x+1$.

Claim: f is surjective

Pf: – fix ~~[illegible]~~. $y \in \mathbb{R}$.
– let $x = \frac{y-1}{2}$

then: $f(x) = f\left(\frac{y-1}{2}\right) = 2\left(\frac{y-1}{2}\right) + 1$
$= y$

– since y was arbitrary, claim is proved.

(xviii) ③ Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

Claim: f is not surjective.

Pf: WTS: $\neg (\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) = y)$

i.e. $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})(f(x) \neq y)$

Let $y = -1$.

Fix $x \in \mathbb{R}$. Then $f(x) = x^2 \geq 0$

hence $f(x) \neq -1 = y$.

Since x was arbitrary: $(\forall x \in \mathbb{R})\ f(x) \neq -1$. ✓

(i.e. $-1 \notin \operatorname{Im} f$).

Injections Def'n: a function $f: A \to B$ is called injective (or one-to-one or (1-1)) iff $(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$

or equivalently: $(\forall x, y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$.

"distinct inputs map to distinct outputs"

ex's – g above is not injective since $1 \neq 3$ but $g(1) = g(3) = *$.

– h, j are injective.

(xix) <u>Proving</u> <u>Injectivity</u>:

<u>Two</u> <u>approaches</u>: fix $x, y \in A$ and either:

① Assume $f(x) = f(y)$ and prove $x = y$

② Assume $x \neq y$ and prove $f(x) \neq f(y)$.

<u>Ex's</u> ① Consider again $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$.

<u>Claim</u> f is injective

<u>Pf</u>: - fix $x, y \in \mathbb{R}$

- assume $f(x) = f(y)$
- i.e. $2x+1 = 2y+1$
- then $2x = 2y$ hence $x = y$ ✓
- Since x, y were arbitrary, claim is proved ✓

② Define $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = n^2$.

<u>Claim</u>: f is <u>injective</u>

<u>Pf</u>: Fix $n, m \in \mathbb{N}$ and sps $n \neq m$.

<u>WTS</u>: $f(n) \neq f(m)$.

<u>Two</u> <u>cases</u>: ① $n < m$

② $m < n$

(xx) if ①: since n, m both positive can square both sides of inequality to get

$n^2 < m^2$ i.e. $f(n) < f(m)$

so in particular $f(n) \neq f(m)$.

if ②: similar ✓

Since n, m were arbitrary, claim is proved. ✓

③ Define $f: \mathbb{Z} \to \mathbb{Z}$ by $f(n) = n^2$

Claim: f is not injective

Pf: $f(-2) = f(2) = 4$

but $-2 \neq 2$. ✓

Bijections Def'n: a function $f: A \to B$ is bijective iff f is both injective and surjective

ex: - g above is not bijective

(surjective, but not injective)

- nor is h

(injective, but not surjective)

- j is bijective

(xxi) <u>Proving bijectivity</u>:

<u>Ex's</u> ① Consider again $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$.

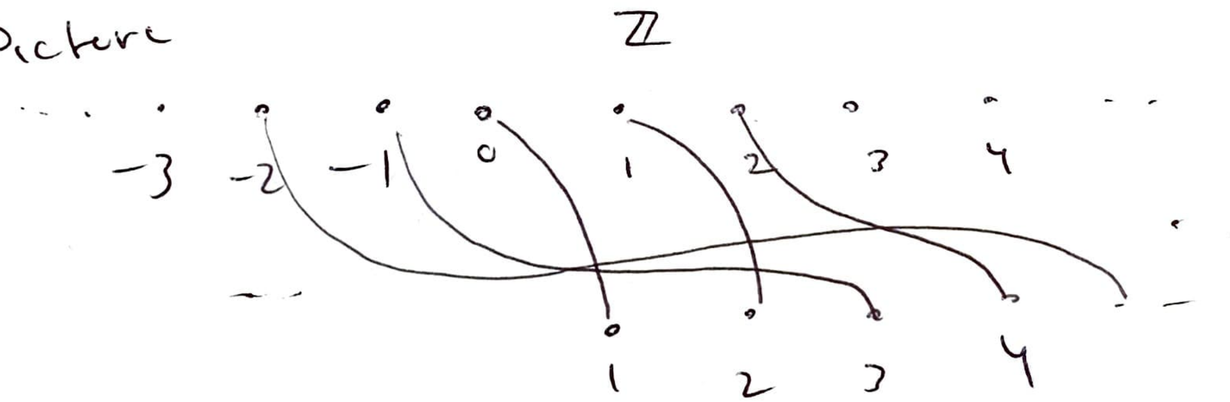
<u>Claim</u>: f is bijective

<u>Pf</u>: we've already showed f is both surjective and injective ✓

② <u>A spicy one</u>: define $f: \mathbb{Z} \to \mathbb{N}$ by

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(-n)+1 & \text{if } n \le 0 \end{cases}$$

Picture



<u>Claim</u> f is bijective

<u>Pf</u>: (surjectivity):

- Fix $n \in \mathbb{N}$
- if n is even, then $n = 2k$ for some $k \in \mathbb{N}$ (so $k > 0$)
- hence $f(k) = 2k = n$

(xxii) - if n is odd then $n = 2k+1$ for some $k \in \mathbb{N}\cup\{0\}$ (hence $k \geq 0$, hence $-k \leq 0$)

- hence $f(-k) = 2k+1 = n$

in either case: $(\exists x \in \mathbb{Z})(f(x) = n)$

- hence f is surjective ✓

(injectivity)

- Fix $n, m \in \mathbb{Z}$ and assume $n \neq m$. We WTS $f(n) \neq f(m)$

- We may assume $n < m$, since case when $m < n$ is similar.

Case 1: $0 < n < m$.

- then $f(n) = 2n < 2m = f(m)$
- hence $f(n) \neq f(m)$

Case 2: $n < m \leq 0$

- then $f(n) = 2(-n) + 1$

$f(m) = 2(-m) + 1$

- observe: since $n < m$

$\Rightarrow -n > -m$

$\Rightarrow 2(-n) + 1 > 2(-m) + 1$

i.e. $f(n) > f(m)$

so that $f(n) \neq f(m)$

Case 3: $n \leq 0 < m$

- then $f(n) = 2(-n) + 1$ is odd

$f(m) = 2m$ is even

(xxiii)

hence $f(n) \neq f(m)$ in this case as well.

↳ hence in all cases $f(n) \neq f(m)$

↳ since n, m were arbitrary, we've proved f is injective. ✓

hence f is bijective ✓