Course overview

- class is an intro. to writing proofs.
- we'll take a tour: cover basic set theory, logic, number theory, combinatorics, topology.

- ... So what is "doing math?"
  ↳ It's not just calculating!
- Roughly: It's about investigating mathematical objects (e.g. integers, right triangles, continuous functions) by proving the truth/falsity of mathematical statements about these objects (e.g. "every continuous function is differentiable").
- mathematical objects are described by precise definitions.

e.g. Def'n A prime number is a positive integer p, such that if n is a positive integer that divides p, then either $n = 1$ or $n = p$.

Not def'ns: - "a line is a flowing point"
- "a point is a place without extension" - Emerson.

↳ suggestive, poetical... but not precise.

- mathematical statements (or propositions) are declarative sentences (concerning mathematical objects) that are either true or false. (i.e. they have a truth value).

e.g. Prop'n 1: (Euclid) There are infinitely many prime numbers. (or in Euclid's words: "There are more primes than found in any list of primes.")

↳ prop'n 1 is true or false: either there are infinitely many primes, or not. (in fact: there are).

↳ establishing truth of a prop'n requires a proof.

- roughly: a sequence of logical deductions from axioms or previously proved statements whose conclusion is the prop'n in question.
- many methods of proof: one is by contradiction.

Proof of prop'n 1: - Sps ("suppose") toward a contradiction that prop'n 1 is false, i.e. that there are only finitely many primes.

- then we can list them as

$$p_1, p_2, \ldots, p_n$$

- consider the integer

$$N = p_1 \cdot p_2 \cdots p_n + 1$$

obtained by multiplying all the primes in our list and adding 1.

e.g. maybe 2, 3, 5, 7 are the only primes.
↓
in this case N would be $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$

- Observe: if we divide N by any of the primes $p_1, \ldots, p_n$ we leave a remainder of 1.

- Hence N is not divisible by any of our primes $p_1, \ldots, p_n$.
- So: either N itself is prime, or there is another prime p not among $p_1, \ldots, p_n$ that divides N.
- either way, there must be another prime not among $p_1, \ldots, p_n$.

a contradiction, as we assumed there were all of the primes.

- Hence our assumption was false.
- Hence there are infinitely many primes.

Sets

- A set is a collection of objects (often defined by a common property)

Cantor: "By a 'set' we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought."

- this is an informal def'n (and in fact a contradictory one)
- formal def'n of set beyond scope of course.
- our approach: we'll write down several fundamental sets that we'll take for granted, then give formal def'ns of operations that allow us to build new sets from old ones.

- sets are enclosed by curly brackets $\{ \cdots \}$.
- objects in a set are called elements.
- $\in$ means "is an element of"
  $\notin$ means "is not an el't of"

Ex's ① Let E denote the set of even positive integers.

- We also write:

$$E = \{2, 4, 6, \ldots\}$$

Then $12 \in E$ but $1 \notin E$ and $-2 \notin E$.

2 - can define finite sets by just writing all of their elements in brackets

- called raster notation
- e.g. if $A = \{2, 4, 6, \pi\}$

$$B = \{\heartsuit, *, \pi\}$$

then $\pi \in A$ and $\pi \in B$
while $\heartsuit \in B$ but $\heartsuit \notin A$.

→ sets are determined by their elements. order, repetition do not matter.

if $A = \{1,2,3\}$
then $A = \{2,1,3\}$
and $A = \{1,2,3,1\}$ as well.

3 - sets can be elements of sets!
- e.g. if $A = \{1,2\}$, $B = \{3,4\}$

then $C = \{A, B\}$
$= \{\{1,2\}, \{3,4\}\}$ is a legit set.

- different from $D = \{1,2,3,4\}$
(C has 2 el'ts, D has 4).

Some fundamental sets

$\mathbb{N} = \{1, 2, 3, \dots\}$ "natural numbers"

$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ "integers"

$\mathbb{Q} = \{ \frac{m}{n} \mid m, n \text{ are in } \mathbb{Z} \text{ and } n \neq 0 \}$ "rational numbers"

$\mathbb{R}$ = set of real ~~nu~~ numbers

$\mathbb{C}$ = set of complex numbers

$= \{ a+bi \mid a, b \text{ are in } \mathbb{R} \}$

So, e.g. we have:

$0 \in \mathbb{Z}$ but $0 \notin \mathbb{N}$

$\frac{22}{7} \in \mathbb{Q}$ but $\frac{22}{7} \notin \mathbb{Z}$.

$\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$

$\sqrt{-1} \to i \in \mathbb{C}$ but $i \notin \mathbb{R}$

- the <u>empty</u> <u>set</u> is the unique set with no elements
- denoted $\emptyset$ or $\{\}$
- <u>not</u> the same as $\{\emptyset\}$
  - this set contains a single element, the empty set contains none.

New sets from old ones

Set-builder notation: given a set X and a well-defined property P, can form a set Y consisting of all $x \in X$ with property P.

We write: $Y = \{x \in X \mid x \text{ has } P\}$ — always need to specify the X where x's being drawn from

or: $Y = \{x \in X \mid P(x)\}$

called "set-builder notation"

Ex's (1) can define $E = \{2, 4, 6, \dots\}$ by:

$$E = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 2\}$$

or, more symbolically

$$E = \{n \in \mathbb{N} \mid \text{there is } k \in \mathbb{N} \text{ s.t. } n = 2 \cdot k\}$$

s.t. = "such that"

(2) once E is defined, can use it to define other sets.

e.g. let

$$O = \{n \in \mathbb{N} \mid \text{there is } k \in E \text{ s.t. } n = k - 1\}$$

$\{1, 3, 5, \dots\}$

③ the set over which you range is important:

$$\{x \in \mathbb{R} \mid x^2 - 2 = 0\} = \{\sqrt{2}, -\sqrt{2}\}$$

whereas $\{x \in \mathbb{Z} \mid x^2 - 2 = 0\} = \emptyset$

since no integers satisfy $x^2 - 2 = 0$.

Some more notation:

- for a given $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$
- e.g. $[5] = \{1, 2, 3, 4, 5\}$.

Subsets:

- a set Y is a subset of X if for every element $y \in Y$ we also have $y \in X$.
- We write: $Y \subseteq X$.

- Y is a proper subset of X if $Y \subseteq X$ but $Y \neq X$.
- we (sometimes) write
$$Y \subsetneq X \quad \text{or} \quad Y \subset X$$
to indicate "Y is a proper subset of X

- whereas $Y \not\subseteq X$ means "Y is not a subset of X".

Ex's (1) $\{1,3\} \subseteq \{1,2,3,4\}$

why: $1 \in \{1,2,3,4\}$ and $3 \in \{1,2,3,4\}$

It is a proper subset so we could write: $\{1,3\} \subsetneq \{1,2,3,4\}$ or $\{1,3\} \subset \{1,2,3,4\}$

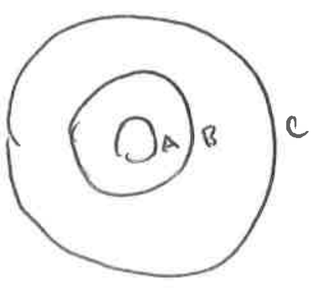
(2) $\{-10,3\} \not\subseteq \{1,2,3,4\}$

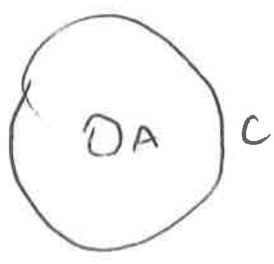
why: $-10 \in \{-10,3\}$ but $-10 \notin \{1,2,3,4\}$.

(3) $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Notice "$\subseteq$" is a transitive relation,

i.e. if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.





Let's prove this from the def'n of $\subseteq$.

Prop'n 1 For any sets A, B, C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Pf: - Sps $x \in A$ is a fixed, arbitrary element of A

- Since $A \subseteq B$, we have $x \in B$, by def'n of $\subseteq$.
- Then, since $B \subseteq C$ we have $x \in C$, again by def'n of $\subseteq$.
- Since $x \in A$ was arbitrary, the same arg. would apply to any el't of A.
- Hence every el't of A is an el't of C, i.e. $A \subseteq C$. ✓

More ex's: (4) For any set X, we have $X \subseteq X$. Pf: fix $x \in X$. Then $x \in X$ too...

(5) Set-builder notation defines a subset, i.e. if $Y = \{x \in X \mid x \text{ has } P\}$ then $Y \subseteq X$.

(6) For any set X, we have $\emptyset \subseteq X$.

→ perhaps unintuitive but here's the

Pf: it is true that if (i) $x \in \emptyset$ then (ii) $x \in X$

simply because (i) never holds!

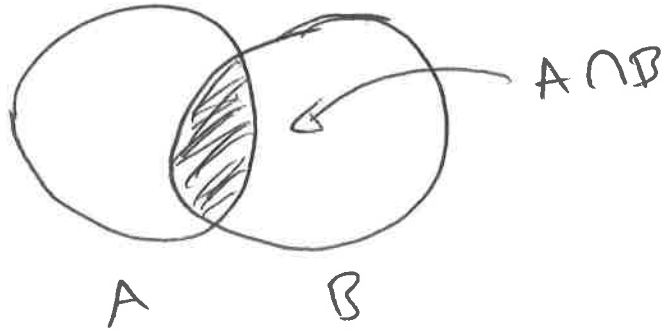
→ more on this type of reasoning later...

Operations on Sets

Intersections: Def'n Given sets A, B the intersection of A and B, denoted $A \cap B$, is the set of el'ts belonging to both A and B.

i.e. $x \in A \cap B$

if (and only if) $x \in A$ and $x \in B$.



Ex ① if $A = \{1,2,3,4\}$ then $A \cap B = \{1,3\}$

$B = \{1,3,5\}$ $\quad A \cap C = \{2,4\}$

$C = \{2,4,6\}$ $\quad B \cap C = \emptyset$.

Def'n Two sets are disjoint iff$^{\text{if and only if}}$ their intersection is $\emptyset$.

ex: B, C above are disjoint.

② **Prop'n**: For any sets A, B we have:

(i) $A \cap B \subseteq A$

(ii) $A \cap B \subseteq B$

↳ "obvious" from the picture but let's practice proving from def'n.

Pf: (i) - Fix $x \in A \cap B$.

- Then by def'n of $\cap$ we have $x \in A$ and $x \in B$.
- Hence in particular, $x \in A$.
- Since x was arbitrary, every el't of $A \cap B$ is an el't of A,

i.e. $A \cap B \subseteq A$. ✓

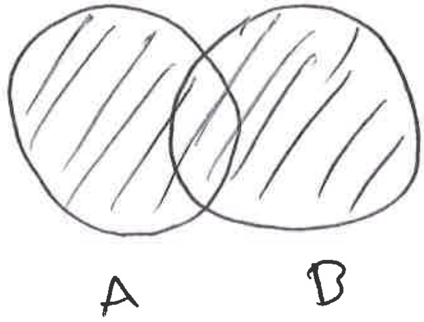
(ii) Similar. ✓

Unions Def'n: the union of A and B, denoted $A \cup B$, is the set of el'ts contained in either A or B.

i.e. $x \in A \cup B$

iff $x \in A$ or $x \in B$.

Note: "or" here (as in all math) is nonexclusive.



$A \cup B$

<u>Ex's</u>: ① $\{1,3,5\} \cup \{2,4,6\} = \{1,2,3,4,5,6\}$
$= [6]$.

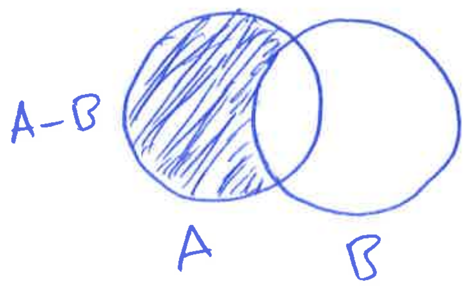
② if $O = \{1,3,5,7 \ldots\}$
$E = \{2,4,6,8, \ldots\}$
then $O \cup E = \mathbb{N} = \{1,2,3,4, \ldots\}$.

③ <u>Prop'n</u>: For any sets A, B we have:
(i) $A \subseteq A \cup B$
(ii) $B \subseteq A \cup B$

Pf: you try.

<u>Difference</u>: the <u>difference</u> of A and B, denoted $A - B$, is the set of el'ts in A that are not in B.

i.e. $x \in A - B$
iff $x \in A$ and $x \notin B$.



Ex: ① if $A = \{1,2,3\}$
$B = \{3,4,5\}$

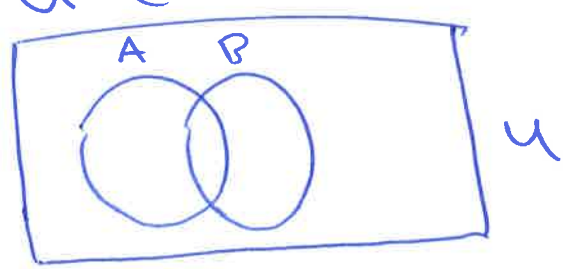
then $A - B = \{1,2\}$
and $B - A = \{4,5\}$.

notice: difference is **not** commutative in general, i.e. $A - B \neq B - A$ in general.

however we always have

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A.$$

Note: in defining $\cap, \cup, -$ it is sometimes convenient to assume that our sets A, B are both subsets of a larger set U (called a universal set)

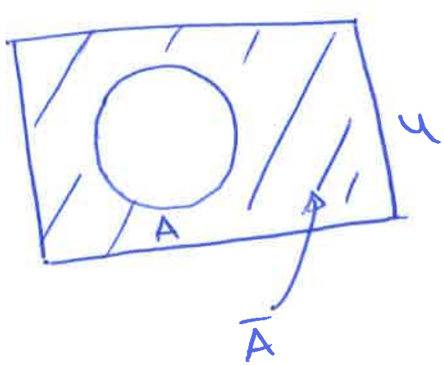


- then we can define these operations using set-builder notation:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}.$$

<u>Complement</u>: <u>Def'n</u> Given a set A, and a universal set U with $A \subseteq U$, the <u>complement</u> of A, denoted $\bar{A}$, is the set of el'ts in U that are not in A.

$$\bar{A} = \{x \in U \mid x \notin A\}$$



<u>note</u>: really $\bar{A}$ is just $U - A$.

<u>Ex</u>: ① Suppose $U = \mathbb{N}$

$$A = \{1,2,3\} = [3]$$
$$E = \{2,4,6,\ldots\}$$
$$O = \{1,3,5,\ldots\}$$

Then: $\bar{A} = \{4, 5, 6, \ldots\}$
$\bar{E} = \{1, 3, 5, \ldots\} = O$
$\bar{O} = \{2, 4, 6, \ldots\} = E$.

Indexing by Sets

- often useful to take unions/intersections of more than two sets
  - ↳ need notation for this.

Ex: - For any $i \in \mathbb{N}$, define

$$A_i = \{-i, 0, i\}$$

so, e.g. $A_1 = \{-1, 0, 1\}$
$A_2 = \{-2, 0, 2\}$ etc...

- Then $A_1 \cup A_2 = \{-2, -1, 0, 1, 2\}$
$A_1 \cup A_2 \cup A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$

or even $A_1 \cup A_2 \cup \cdots \cup A_{10} = \{-10, -9, \ldots, 8, 9, 10\}$

- we could denote above union as $\bigcup_{i=1}^{10} A_i$

- but instead we'll think of the index variable i as "ranging over" the set $[10] = \{1, 2, \ldots, 10\}$ and write

$$\bigcup_{i \in [10]} A_i$$

- in the same way we could write

$$\bigcup_{i \in \{1,2\}} A_i \quad \text{for} \quad A_1 \cup A_2, \quad \text{and}$$

$$\bigcup_{i \in \{1,2,3\}} A_i \quad \text{for} \quad A_1 \cup A_2 \cup A_3$$

More generally:

<u>Def'n</u> Sps I is a set (called the <u>index set</u>) s.t. for every $i \in I$ we have defined a set A_i. We define

$$\bigcup_{i \in I} A_i$$

as the set of el'ts contained in <u>at least one</u> of the A_i's.

i.e. $x \in \bigcup_{i \in I} A_i$

iff there exists an i s.t. $x \in A_i$

We also define:

$\bigcap_{i \in I} A_i$ as the set of el'ts contained in every A_i

i.e. $x \in \bigcap_{i \in I} A_i$ iff $x \in A_i$ for every $i \in I$

Ex's For $i \in \mathbb{N}$, define $A_i = \{-i, 0, i\}$ as before.

① Let $I = [10] = \{1, 2, \dots, 10\}$

Then $\bigcup_{i \in I} A_i = \bigcup_{i \in \{1,2,\dots,10\}} A_i = \{-10, -9, \dots, 8, 9, 10\}$

whereas $\bigcap_{i \in \{1,2,\dots,10\}} A_i = \{0\}$.

② An infinite union:

$$\bigcup_{i \in \mathbb{N}} A_i = A_1 \cup A_2 \cup \dots$$
$$= \{\dots, -2, -1, 0, 1, 2, \dots\}$$
$$= \mathbb{Z}.$$

③ Let $E = \{2, 4, 6, \dots\}$

Then $\bigcup_{i \in E} A_i = \{\dots -4, -2, 0, 2, 4, \dots\}$

④ It may be that the indices themselves are sets!

e.g. let

$$X = \{\{1,2\}, \{1,3\}, \{1,4\}\}$$

what is

$$\bigcup_{y \in X} y \ ?$$

The union of all sets in X!

$$\bigcup_{y \in X} y = \{1,2\} \cup \{1,3\} \cup \{1,4\}$$

$$= \{1,2,3,4\}$$

Equality of Sets

- a set is determined by its elements: two sets are equal exactly when they have the same el'ts
- can make this a precise def'n.

Def'n For any sets A, B we define $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

i.e. $A = B$

iff whenever $a \in A$ then $\cancel{a} \in B$
and whenever $b \in B$ then $b \in A$.

↳ main import of def'n is in proofs: to prove $A = B$ one proves

(i) $A \subseteq B$

(ii) $B \subseteq A$.

↳ more on this later...