

Homework #9

1. Let \mathcal{F} denote the set of *all* functions from \mathbb{N} to \mathbb{N} , that is, $\mathcal{F} = \{f \subseteq \mathbb{N} \times \mathbb{N} \mid f \text{ is a function}\}$. Define a relation R on \mathcal{F} by the rule $(f, g) \in R$ iff for every $n \in \mathbb{N}$ we have $f(n) \leq g(n)$. Prove that R is a partial order on \mathcal{F} .
2. Fix $m, n \in \mathbb{N}$. Define a mapping $f : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ by $f([a]_n) = [a]_m$.
 - a. Prove that if $m \mid n$ then f is a well-defined function. That is, prove that if $[a]_n = [b]_n$ then $f([a]_n) = f([b]_n)$.
 - b. Let $n = 12$ and $m = 3$. Write $\text{PreIm}_f(\{[1]_3, [2]_3\})$ in roster notation.
 - c. Suppose $m \nmid n$. Show that f is ill-defined. That is, show there exist $a, b \in \mathbb{Z}$ such that $[a]_n = [b]_n$ but $f([a]_n) \neq f([b]_n)$.
3. Suppose that A, B , and C are nonempty sets and $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions.
 - a. Prove that if f and g are surjections then so is $g \circ f$.
 - b. Prove that if f and g are injections then so is $g \circ f$.
 - c. Use your results from parts (a.) and (b.) to prove that if f and g are bijections then so is $g \circ f$.
4. Suppose X and Y are nonempty sets and $f : X \rightarrow Y$ is a function. Define a new function $F : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ by $F(B) = \text{PreIm}_f(B)$. Prove that F is injective if and only if f is surjective.