Homework #7

1. Let R be a relation defined on $\mathcal{P}(\mathbb{Z})$ defined by

$$(A, B) \in R \text{ if and only if } A \cap B \neq \emptyset.$$

Prove or disprove each of the following statements:

a. R is reflexive.

b. R is symmetric.

c. R is transitive.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function on $\mathbb{R}$. Define a relation R_f on $\mathbb{R}$ by the rule $(x, y) \in R_f$ if and only if $f(x) = f(y)$. Explicitly, we have $R = \{(x, y) \in \mathbb{R}^2 \mid f(x) = f(y)\}$.

a. Prove that R_f is an equivalence relation.

b. Suppose that f is the squaring function defined by $f(x) = x^2$. For a fixed real number $r \in \mathbb{R}$, determine the equivalence class $[r]_{R_f}$.

3. (Contructing the rationals) Define a relation $\sim$ on $\mathbb{Z} \times \mathbb{N}$ such that for any two pairs $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{N}$ we have:

$$(a, b) \sim (c, d) \text{ if and only if } ad = bc$$

a. Prove that $\sim$ is an equivalence relation

b. Determine the set $[(0, 3)]_\sim$ and write it using set-builder notation.

c. Write out three elements of $[(2, 5)]_\sim$.

d. We can naturally identify $(\mathbb{Z} \times \mathbb{N})/\sim$ with one of our standard sets. Which set is this?

4. (Modular arithmetic) A key property of the relation of congruence modulo n is that it is preserved by addition and multiplication. In this sense, congruence behaves like equality. For example, from the relation $2 \equiv 5 \pmod 3$ we can, by adding 13 to both sides, deduce $15 \equiv 18 \pmod 3$. And by multiplying both sides by 2 we obtain $4 \equiv 10 \pmod 3$.

Prove that this works in general. That is, fix $n \in \mathbb{N}$ and prove that for any $x, y, k \in \mathbb{Z}$ we have

a. if $x \equiv y \pmod n$, then $k + x \equiv k + y \pmod n$

b. if $x \equiv y \pmod n$, then $kx \equiv ky \pmod n$

5. A relation R on a set A is called *irreflexive* iff $(\forall x \in A)((x, x) \notin R)$. It is called *asymmetric* iff $(\forall x, y \in A)((x, y) \in R \Rightarrow (y, x) \notin R)$.

Prove that a relation R on a set A is asymmetric iff it is both irreflexive and anti-symmetric.