Homework #4

1. Consider the following variable propositions:

   Let $P(x)$ be the proposition "$1 \leq x \leq 3$"
   Let $Q(x)$ be the proposition "$(\exists k \in \mathbb{Z})(x = 2k)$"
   Let $R(x)$ be the proposition "$x^2 = 4$"

   Recall that a statement is in *positive form* if the only negation symbols in the statement appear next to substatements that do not contain quantifiers or connectives.

   For each of the following statements, write the negation in a logically equivalent positive form. Then decide which claim (the original or the negation) is true (no proof required).

   a.) $(\forall x \in \mathbb{Z})(P(x) \Rightarrow Q(x))$
   b.) $(\exists x \in \mathbb{Z})(R(x) \wedge P(x))$
   c.) $(\forall x \in \mathbb{Z})(R(x) \Rightarrow P(x))$
   d.) $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x \neq y \wedge P(x) \wedge Q(x))$

2. Consider the following propositions, which assert that the rational numbers are *dense*, and the integers are *discrete*, respectively:

   (a) *Strictly between any two distinct rational numbers lies a third rational number.*
   (b) *For every integer n, there is a strictly larger integer m, such that there are no integers strictly between n and m.*

   Write out these propositions symbolically, using only logical symbols and the sets $\mathbb{Q}$ and $\mathbb{Z}$.

3. For every $i \in \mathbb{N}$, define a set $A_i \subseteq \mathbb{N}$ such that the indexed family of sets $\{A_i : i \in \mathbb{N}\}$ satisfies *all* of the following properties (recall that "$\subsetneq$" means "is a *strict* subset of"):

   a.) $(\forall n \in \mathbb{N})(\exists i \in \mathbb{N})(n \in A_i)$
   b.) $(\forall i \in \mathbb{N})(\exists n \in \mathbb{N})(n \notin A_i)$
   c.) $(\forall i, m \in \mathbb{N})(\exists n \in \mathbb{N})(n > m \wedge n \in A_i)$
   d.) $(\exists j \in \mathbb{N})(\forall i \in \mathbb{N})(i \neq j \Rightarrow A_j \subsetneq A_i)$

   Then, *prove* that the family you've defined satisfies each of these properties.

4. Use a chain of logical equivalences to prove the following propositions.

   a.) Given a universal set U and sets $A, B \subseteq U$, it is the case that $(A \cup B) \cap \overline{A} = B - A$.
   b.) For all sets A, B, and C, it is the case that $A \cap (B - C) = (A \cap B) - (A \cap C)$.

   (A possibly helpful *hint*: if P is a false statement, then $P \vee Q$ is logically equivalent to Q.)

5. Consider the following proposition:

   For all integers n, n is an integer multiple of 3 if and only if $n^2 - 1$ is not a multiple of 3.

   a.) Write out this proposition symbolically, using only logical symbols and the set $\mathbb{Z}$.
   b.) Prove the proposition. (You should be able to prove it using nothing more than the definition of being a multiple of 3, and the fact that every integer has a remainder of 0, 1, or 2 when divided by 3.)