Homework #12

1. Let $C = \{f \subseteq \mathbb{R}^2 \,|\, f : \mathbb{R} \to \mathbb{R} \text{ is a function}\}$. Prove that $|\mathbb{R}| < |C|$ by showing there is no surjection $F : \mathbb{R} \to C$.

2. Fix $a, b \in \mathbb{Z}$, not both 0, and $m \in \mathbb{Z}$. Prove that $gcd(a, b) = \gcd(a + bm, b)$ by proving $gcd(a, b) \leq \gcd(a + bm, b)$ and $gcd(a, b) \geq \gcd(a + bm, b)$.

3. Fix $a, b \in \mathbb{Z}$, not both 0, and $m \in \mathbb{N}$. Prove that $\gcd(am, bm) = m \gcd(a, b)$.

   (One approach: prove $\gcd(am, bm) \leq m \gcd(a, b)$ using Bezout's theorem, and $\gcd(am, bm) \geq m \gcd(a, b)$ directly.)

4. Fix $a, b \in \mathbb{Z}$ and suppose that $\gcd(a, b) = 1$. Prove that $\gcd(a + b, a - b)$ is either 1 or 2.

5. Fix $p \in \mathbb{N}$ a prime, with $p > 3$. Prove that $p^2 \equiv 1 \pmod{24}$.