Homework #11

Note: there is an extra credit question on the second page.

1. Suppose that $f : A \to B$ is a function, viewed as a relation (set of ordered pairs). Let f^* denote the "reverse" relation: $f^* = \{(b, a) \in B \times A \,|\, (a, b) \in f\}$

   For example, if $f : \{1, 2, 3\} \to \{\star, \heartsuit, \triangle\}$ is the function

   $$f = \{(1, \star), (2, \heartsuit), (3, \star)\},$$

   then

   $$f^* = \{(\star, 1), (\heartsuit, 2), (\star, 3)\}.$$

   We proved in class that if f is a bijection, then f^* is also a bijection (from B back onto A). In general, f^* need not even be a function, as we can see from the example above. In fact, f^* is a function if and only if f is a bijection, as you will prove below.

   (a) Prove that if a function $f : A \to B$ is *not* a bijection, then f^* is *not* a function. (To do this, prove that if f is not an injection, then f^* is not a function, and also prove that if f is not a surjection, then f^* is not a function.)

   (b) Consider the function $f : \{1, 2, 3\} \to \{1, 2, 3, 4, 5\}$ defined the $f(n) = n + 1$. Observe that this function is injective, but not surjective.

   Define a surjection $g : \{1, 2, 3, 4, 5\} \to \{1, 2, 3\}$ such that $g \supseteq f^*$. (In class we proved such a surjection always exists.)

   (c) Consider the function $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ defined by $f(n) = |n|$. Observe that this function is surjective, but not injective.

   Define an injection $g : \mathbb{N} \to \mathbb{Z}$ such that $g \subseteq f^*$. (In class we proved such an injection always exists.)

2. Suppose that $\{A_n : n \in \mathbb{N}\}$ is a family of sets (not necessarily pairwise disjoint) such that A_n is countable for every n. Prove that $\bigcup_{n\in\mathbb{N}} A_n$ is countable. (Hint: prove that there is a surjection $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{n\in\mathbb{N}} A_n$ and say why this is sufficient.)

3. Show that there is a bijection $f : [0, 1) \to [0, 1]$ by defining injections $g : [0, 1) \to [0, 1]$ and $h : [0, 1] \to [0, 1)$. Prove that that the functions h and g that you define are injections. (Can you see how to define such a bijection f directly?)

4. (Representing subsets of $\mathbb{N}$ as infinite strings) Consider the set $B = \{f \subseteq \mathbb{N} \times \{0, 1\} \,|\, f : \mathbb{N} \to \{0, 1\}$ is a function$\}$. Construct a bijection $F : \mathcal{P}(\mathbb{N}) \to B$. Prove that the function F you construct is a bijection.

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("A vanishing abundance") At 11am you find yourself next to an infinite pile of balls and a very large bucket. You take 10 of the balls from the pile and place them in the bucket. Then immediately you remove one ball and throw it away, so that 9 balls remain in the bucket.

At 11:30 you repeat the process, taking 10 new balls from the pile, placing them in the bucket, and then removing one and throwing it away. Now there are 18 balls in the bucket.

Once again at 11:45 you take 10 balls from the pile, put them in the bucket, then take one out and throw it away. Now there are 27 balls in the bucket.

And so on, so that whenever the time before noon halves from the previous placement, you put ten more balls from the pile into the bucket, and then take one out and throw it away.

How many balls are in the bucket at noon?

* * *

XC. (1 point) The answer actually depends on *which* balls you take out and throw away at every step. Show that it is possible to arrange that the number of balls remaining in the bucket at noon is

(a) infinite,

(b) 0,

(c) 12.