Functions

- functions, like relations, are ubiquitous in math.
- But what "are" functions?
- intuitively: a rule that assigns to each x in a domain A a unique output $f(x)$ in a codomain B.

→ can define functions rigorously as a special type of relation

Def'n a function (with domain A and codomain B) is a relation $f \subseteq A \times B$ such that
for every $a \in A$
there is a unique $b \in B$
s.t. $(a,b) \in f$

i.e.
$$(\forall a \in A)(\exists b \in B)\left[(a,b) \in f \wedge (\forall c \in B)((a,c) \in f \Rightarrow b = c)\right]$$

→ we write
$f : A \to B$
to indicate that a subset
$f \subseteq A \times B$
is a function

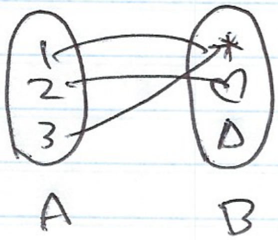
→ we also write
$f(a) = b$
to mean $(a,b) \in f$

<u>Note</u>: - def'n says every $a \in A$ is assigned an output $f(a) \in B$

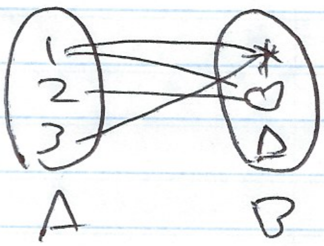
- does <u>not</u> insist that for every $b \in B$ there is $a \in A$ s.t. $f(a) = b$ (functions w/ this property are called <u>onto</u>)

<u>ex</u>: 1 Let $A = \{1, 2, 3\}$
$B = \{*, \heartsuit, \Delta\}$

then $f = \{(1, *), (2, \heartsuit), (3, *)\}$ is a function from A to B

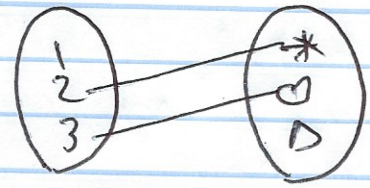


but $g = \{(1, *), (1, \heartsuit), (2, \heartsuit), (3, *)\}$ is <u>not</u> a function since 1 does not have a unique output



not a function

<u>nor</u> <u>is</u> $h = \{(2, *), (3, \heartsuit)\}$ since 1 is not assigned an output



not a function

② we'll often define functions by some rule, e.g.

$$f: \mathbb{R} \to \mathbb{R}$$
$$f(x) = x^2$$

or

$$f: \mathbb{R} \to \mathbb{Z}$$
$$f(x) = \lfloor x \rfloor$$

but behind the scenes we still consider these f's to be <u>sets</u> of ordered pairs

e.g. if we define $f(x) = x^2$
then $(2,4) \in f$
$(3,9) \in f$
$(4,5) \notin f$

<u>Warning</u>: not all rules yield well-defined functions.

e.g. suppose we "define"
$f: \mathbb{Q} \to \mathbb{Z}$
by the rule
$f(m/n) = m+n$

then this "function" is not one

$$f(1/2) = 1+2 = 3 \neq 6 = 2+4 = f(2/4)$$

but $1/2 = 2/4$.

- so f assigns multiple outputs to the same input.
- what's going on? really there's an implicit equiv. relation on fractions ($\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots$)
- our rule defines f on a representative of an equiv class

- In general: when given a rule "defining" some $f \subseteq A \times B$ to verify f is a function one must show
  1. $\forall a \in A \; \exists b \in B$ s.t. $(a,b) \in f$
  2. if $a = a'$ then $f(a) = f(a')$

Equality of Functions

Q: what does it mean for functions $f: A \to B$ and $g: A \to B$ to be equal?

A: $f = g$ iff they're equal as sets of ordered pairs, i.e. iff $f \subseteq g$ and $g \subseteq f$.

equivalently, $f = g$ iff $(a,b) \in f \Leftrightarrow (a,b) \in g$.

In practice, easier to use following:

Theorem: if $f: A \to B$ and $g: A \to B$ are functions then $f = g$ iff
$$(\forall a \in A)(f(a) = g(a)).$$

Pf: you try.

The point: functions can be equal despite being defined by different rules.

ex: let $A = \{1, 2, 3\}$
define $f: A \to \mathbb{N}$
$g: A \to \mathbb{N}$
by
$$f(x) = x^3 + 11x$$
$$g(x) = 6x^2 + 6$$

Then
$$f(1) = 12 = g(1)$$
$$f(2) = 30 = g(2)$$
$$f(3) = 60 = g(3)$$

i.e. $f = \{(1,12), (2,30), (3,60)\} = g$.

(what's the magic trick?
$$f - g = x^3 - 6x^2 + 11x - 6$$
$$= (x-1)(x-2)(x-3)$$
)

Images

Def'n Suppose $f: A \to B$ is a function and $X \subseteq A$.
The *image of X under f*, denoted $Im_f(X)$, is defined as:

$$Im_f(X) = \{ b \in B \mid (\exists a \in X)\, f(a) = b \}$$

the pt: $b \in Im_f(X)$ iff $\exists x \in X$ s.t. $f(x) = b$

more informally we write:

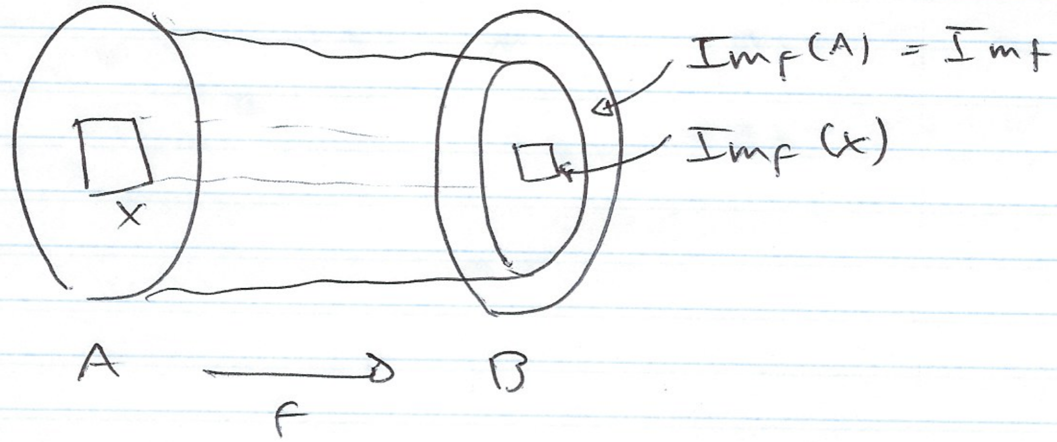
$$= \{ f(a) \mid a \in X \}.$$

When $X = A$ we just say that $Im_f(A)$ is the *image of f* and sometimes just write Im_f

Def'n says: - $Im_f(X)$ is the "set of outputs of el'ts in X"
- $Im_f = Im_f(A)$ is the "set of all outputs."

→ in particular: if $x \in X$ then $f(x) \in Im_f(X)$

Picture:



Ex: ① Let $A = \{1, 2, 3\}$
$B = \{*, ♡, \Delta\}$
$f = \{(1, *), (2, ♡), (3, *)\}$

Then: - $Im_f(\{1,3\}) = \{f(1), f(3)\}$
$= \{*, *\}$
$= \{*\}$

- $Im_f = Im_f(A) = \{f(1), f(2), f(3)\}$
$= \{*, ♡, *\}$
$= \{*, ♡\}$.

② Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

Then: - $Im_f(\{-1, 0, 1\})$
$= \{(-1)^2, 0^2, 1^2\}$
$= \{0, 1\}$

- $Im_f = \{x \in \mathbb{R} \mid x \geq 0\}$.

Functions add a layer of complexity to basic set theory of $\cap, \cup, \ldots$ we studied earlier.

Prop'n Suppose $f: A \to B$ is a function and $S, T \subseteq A$.
Then:
$Im_f(S \cap T) \subseteq Im_f(S) \cap Im_f(T)$

<u>Pf</u>: - fix $y \in Im_f(S \cap T)$
- then $\exists x \in S \cap T$ s.t. $f(x) = y$
- hence $x \in S$ and $x \in T$
- hence $f(x) \in Im_f(S)$ <u>and</u> $f(x) \in Im_f(T)$
- i.e. $y \in Im_f(S)$ and $y \in Im_f(T)$
- i.e. $y \in Im_f(S) \cap Im_f(T)$

since y was arbitrary the prop'n is proved.

<u>Note</u>: in general we <u>don't</u> have
$Im_f(S \cap T) \supseteq Im_f(S) \cap Im_f(T)$

<u>e.g.</u> consider $f(x) = x^2$ on $\mathbb{R}$.
let $S = \{-1, 0\}$ $T = \{0, 1, 2\}$

<u>then</u>:

$$Im_f(S) = \{f(-1), f(0)\} = \{1, 0\}$$

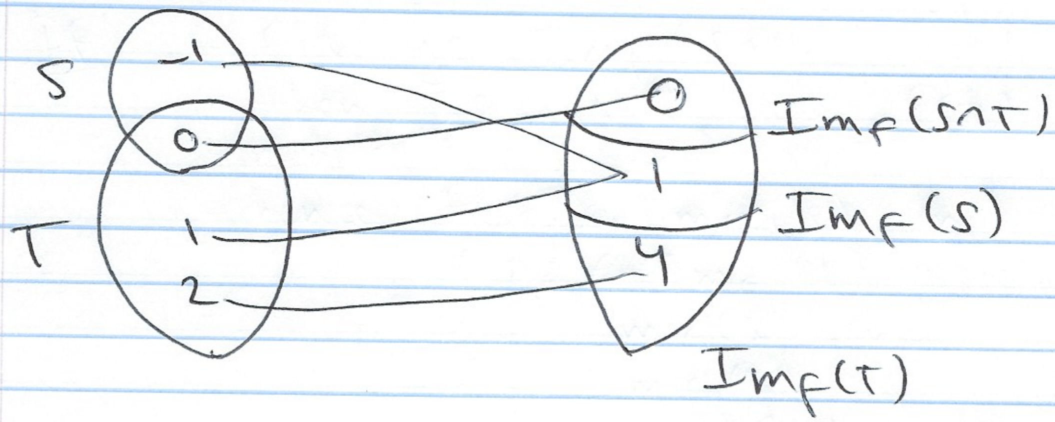
$$Im_f(T) = \{f(0), f(1), f(2)\} = \{0, 1, 4\}$$

hence $Im_f(S) \cap Im_f(T) = \{0, 1\}$

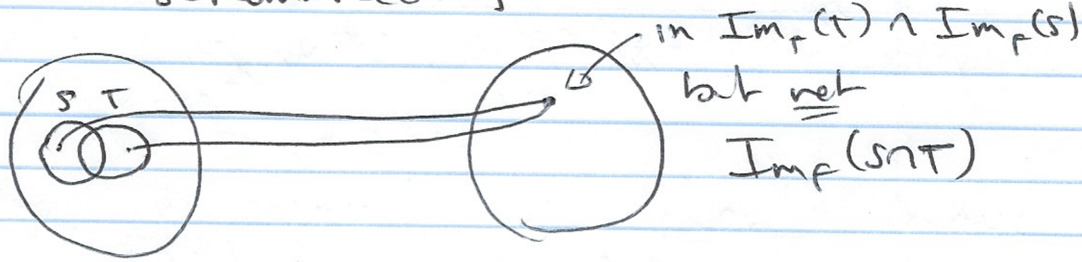
<u>but</u>: $Im_f(S \cap T) = Im_f(\{0\}) = \{f(0)\} = \{0\}$

So in this case
$Im_f(S \cap T) \subsetneq Im_f(S) \cap Im_f(T)$.

$\hookrightarrow$ the essence: function can send multiple inputs to same output!



more schematically:

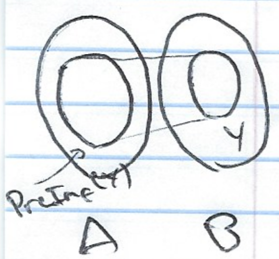


Preimages

Def'n: Sps $f: A \to B$ is a function and $Y \subseteq B$. The preimage of Y under f, denoted $PreIm_f(Y)$, is defined as:

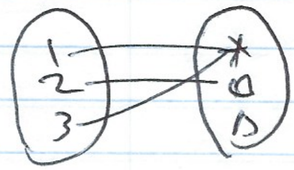
$$PreIm_f(Y) = \{x \in A \mid f(x) \in Y\}$$

= "stuff that lands in Y when we hit it with f"



Note: Since $f(x) \in B$ for every $x \in A$, we don't separately define $PreIm_f(B)$, since this is always just A.

ex: (1) $A = \{1, 2, 3\}$
$B = \{*, \square, \Delta\}$
$f = \{(1,*), (2,\square), (3,*)\}$



then: $\operatorname{PreIm}_f(\{*\}) = \{x \in A \mid f(x) \in \{*\}\}$
$= \{x \in A \mid f(x) = *\}$
$= \{1,3\}$

$\operatorname{PreIm}_f(\{*,\square\}) = \{x \in A \mid f(x) \in \{*,\square\}\}$
$= \{1,2,3\} = A$

$\operatorname{PreIm}_f(\{\Delta\}) = \{x \in A \mid f(x) \in \{\Delta\}\}$
$= \emptyset$.

(2) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by
$f(x) = x^2$

Then $\operatorname{PreIm}_f(\{0,1\})$
$= \{x \in \mathbb{R} \mid f(x) \in \{0,1\}\}$
$= \{x \in \mathbb{R} \mid x^2 \in \{0,1\}\}$
$= \{-1, 0, 1\}$.

$\operatorname{PreIm}_f([0,2])$
$= \{x \in \mathbb{R} \mid x^2 \in [0,2]\}$
$= \{x \in \mathbb{R} \mid 0 \le x^2 \le 2\}$
$= \{x \in \mathbb{R} \mid x^2 \le 2\}$
$= \{x \in \mathbb{R} \mid -\sqrt{2} \le x \le \sqrt{2}\}$
$= [-\sqrt{2}, \sqrt{2}]$

$$\mathrm{PreIm}_f([0,\infty)) = \{x \in \mathbb{R} \mid x^2 \in [0,\infty)\} = \mathbb{R}.$$

$$\mathrm{PreIm}_f(\mathbb{R}) = \mathbb{R}.$$

Q: What if we take the preimage of the image of $X \subseteq A$? or the image of the preimage of $Y \subseteq B$?

<u>Prop'n</u>: Suppose $f: A \to B$ is a function.

(i) Fix $X \subseteq A$

<u>Then</u> $\mathrm{PreIm}_f(\mathrm{Im}_f(X)) \supseteq X$

(ii) Fix $Y \subseteq B$

<u>Then</u> $\mathrm{Im}_f(\mathrm{PreIm}_f(Y)) \subseteq Y$.

<u>Pf</u>: (i) Fix $x \in X$.

By def'n: $\mathrm{PreIm}_f(\mathrm{Im}_f(X))$
$= \{y \in A \mid f(y) \in \mathrm{Im}_f(X)\}$

Now, we know $f(x) \in \mathrm{Im}_f(X)$ by def'n of $\mathrm{Im}_f(X)$

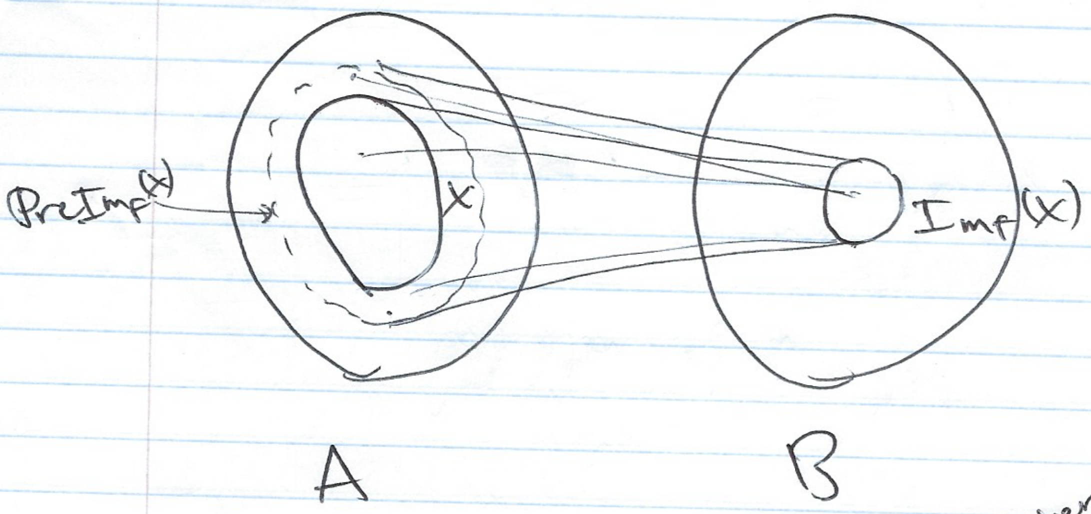
Hence $x \in \mathrm{PreIm}_f(\mathrm{Im}_f(X))$
Since x was arbitrary, (i) is proved ✓

(ii) Fix $y \in Im_f(PreIm_f(Y))$
then $\exists z \in PreIm_f(Y)$
s.t. $f(z) = y$

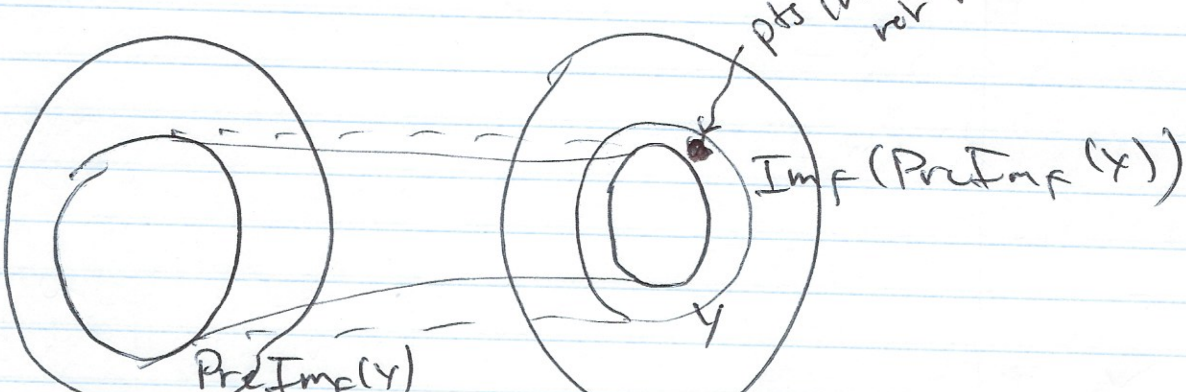
but by def'n $PreIm_f(Y) = \{x \in A \mid f(x) \in Y\}$
hence $f(z) \in Y$
~~[illegible]~~ i.e. $y \in Y$
Since y was arbitrary
(ii) is proved ✓

Pitchas:

(i)



(ii)



<u>In general</u>: neither containment can be reversed.

(i) e.g. Let $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
Let $X = \{1\}$

<u>Then</u>: $\mathrm{Im}_f(X) = \mathrm{Im}_f(\{1\})$
$= \{f(1)\}$
$= \{1\}$

$\Rightarrow \mathrm{PreIm}_f(\mathrm{Im}_f(X))$
$= \mathrm{PreIm}_f(\{1\}) = \{x \in \mathbb{R} \mid x^2 = 1\}$
$= \{-1, 1\} \neq X.$

hence $X \subsetneq \mathrm{PreIm}_f(\mathrm{Im}_f(X))$ in this case.

(ii) Let $Y = \{-5, 1\}$
Then $\mathrm{PreIm}_f(Y)$
$= \{x \in \mathbb{R} \mid f(x) \in \{-5, 1\}\}$
$= \{x \in \mathbb{R} \mid x^2 \in \{-5, 1\}\}$
$= \{-1, 1\}$

hence $\mathrm{Im}_f(\mathrm{PreIm}_f(Y))$

$= \mathrm{Im}_f(\{-1, 1\})$
$= \{f(-1), f(1)\}$
$= \{1\} \subsetneq Y.$

(ii)

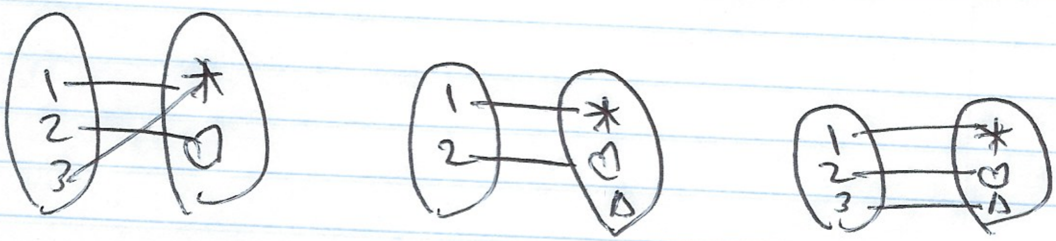
–jections

let $A = \{1, 2, 3\}$
$B = \{*, ♡\}$

$C = \{1, 2\}$
$D = \{*, ♡, \Delta\}$

define: $g : A \to B$
$h : C \to D$
$j : A \to D$

by: $g = \{(1,*), (2,♡), (3,*)\}$
$h = \{(1,*), (2,♡)\}$
$j = \{(1,*), (2,♡), (3,\Delta)\}$



g h J

Surjections:

Def'n: a function $f : A \to B$ is surjective (or onto) iff $\mathrm{Im}\, f = B$ that is, iff

$$(\forall b \in B)(\exists a \in A)(f(a) = b)$$

(ii)

ex's
- g and j above are surjections
- h is not because $\Delta \notin \text{Im}_h$

Proving surjectivity

ex: ① Define $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$
by $f((m,n)) = m+n$.

Claim: f is surjective

Pf: WTS $(\forall x \in \mathbb{Z})(\exists (m,n) \in \mathbb{Z}\times\mathbb{Z})$
$f(m,n) = x$

- So fix $x \in \mathbb{Z}$
- observe $f(0,x) = 0+x = x$
- hence $\exists (m,n)$ s.t. $f(m,n) = x$
namely $(m,n) = (0,x)$.
- since x was arbitrary, claim is proved.

② Define $f: \mathbb{R} \to \mathbb{R}$ by
$f(x) = 2x+1$

Claim f is surjective.

Pf: - Fix $y \in \mathbb{R}$
- let $x = \frac{y-1}{2}$
- then
$f(x) = 2\left(\frac{y-1}{2}\right) + 1 = y-1+1 = y$
- since y was arbitrary claim is proved.

(iii)

③ Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$

<u>Claim</u> f is <u>not</u> surjective

<u>Pf.</u> WTS: $\neg\left((\forall y \in \mathbb{R})(\exists x \in \mathbb{R})(f(x) = y)\right)$

i.e. $(\exists y \in \mathbb{R})(\forall x \in \mathbb{R})\ f(x) \neq y$

Let $y = -1$

Then $\forall x \in \mathbb{R}$, $f(x) = x^2 \geq 0$

hence $f(x) \neq -1$ ✓

<u>Injections</u>

<u>Def'n</u>: a function $f: A \to B$ is called <u>injective</u> (or one-to-one or 1-1)

iff

$$(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$$

↳ sometimes helpful to write def'n in contrapositive form;

$$(\forall x, y \in A)(x \neq y \Rightarrow f(x) \neq f(y))$$

"distinct inputs map to distinct outputs"

<u>ex</u>: - g above is <u>not</u> injective since $1 \neq 3$ but $g(1) = g(3) = *$

- h, j <u>are</u> injective.

(iv)

Proving injectivity

Two approaches: Fix $x, y \in A$ and either:

① Assume $f(x) = f(y)$, prove $x = y$
② Assume $x \neq y$, prove $f(x) \neq f(y)$.

Ex's ① Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 5x + 6$

Claim f is injective

Pf: - Fix $x, y \in \mathbb{R}$
- assume $f(x) = f(y)$
i.e. $5x + 6 = 5y + 6$
- then $5x = 5y$
- hence $x = y$

Since x, y were arbitrary, claim is proved.

② Define $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = n^2$

Claim f is injective

Pf: Fix $n, m \in \mathbb{N}$ and assume $n \neq m$.
(WTS: $f(n) \neq f(m)$)
Two cases: (i) $n < m$
(ii) $m < n$

if (i): since n, m both positive we may square both sides

(v)

of inequality to get
$n^2 < m^2$
i.e. $f(n) < f(m)$
hence $f(n) \neq f(m)$

(ii) similar

↳ since n, m were arbitrary claim is proved ✓

③ Define $f: \mathbb{Z} \to \mathbb{Z}$ $f(n) = n^2$

Claim: f is not injective

Pf: - $f(-2) = f(2) = 4$
- but $-2 \neq 2$
- hence f is not injective.

Bijections

Def'n a function $f: A \to B$ is bijective iff it is both injective and surjective

ex: - g is not a bijection (surjective, but not injective)
- nor is h (injective but not surjective)
- j is bijective
i.e. is a bijection

Proving bijectivity

ex ① Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x - 1$

Claim f is a bijection (i.e. is bijective)

Pf: (surjectivity): - fix $y \in \mathbb{R}$
- let $x = y + 1/3$
- then $f(x) = 3(y+1/3) - 1$
$= y + 1 - 1$
$= y$
- since y was arbitrary, f is surjective ✓

(injectivity) - fix $x, y \in \mathbb{R}$ assume $f(x) = f(y)$
i.e. $3x - 1 = 3y - 1$
then $3x = 3y$
i.e. $x = y$ ✓
since x, y arbitrary, f is injective ✓

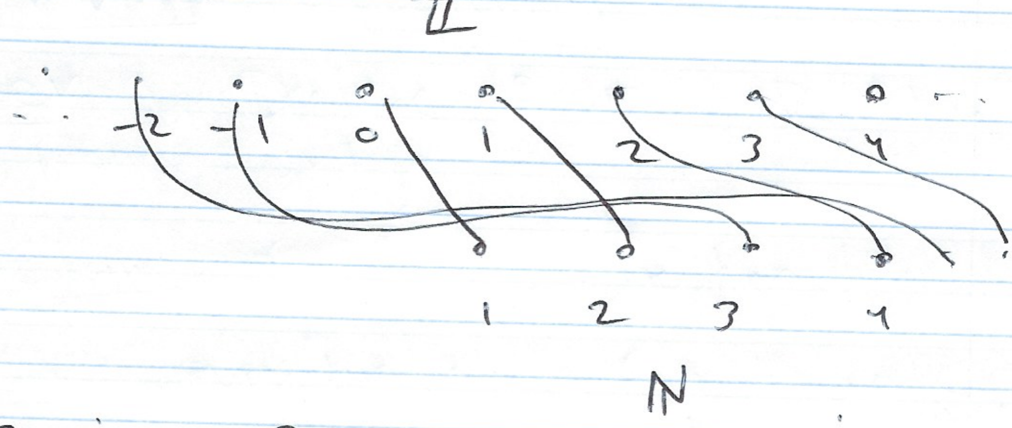
hence f is bijective, as claimed.

② Define $f: \mathbb{Z} \to \mathbb{N}$ by:

$$f(n) = \begin{cases} 2n & \text{if } n > 0 \\ 2(-n)+1 & \text{if } n \leq 0 \end{cases}$$

(vii)

Picture



Claim: f is a bijection

Pf: (surjectivity):

- Fix $n \in \mathbb{N}$
- if n is even, then $n = 2k$ for some $k \in \mathbb{N}$ (hence $k > 0$)
- hence $f(k) = 2k = n$
- if n is odd then $n = 2k+1$ for some $k \in \mathbb{N} \cup \{0\}$
- hence $-k \leq 0$
- hence $f(-k) = 2k+1 = n$

↳ In either case $(\exists x \in \mathbb{Z})\ f(x) = n$

- hence f is surjective ✓

(injectivity):

- Fix $n, m \in \mathbb{Z}$ and assume $n \neq m$.
- We may assume $n < m$ since if $m < n$ the argument is similar.

(viii)

Case 1: $0 < n < m$

- then $f(n) = 2n < 2m = f(m)$
- hence $f(n) \neq f(m)$ ✓

Case 2: $n < m \leq 0$

~~[illegible]~~

- then $f(n) = 2(-n) + 1$
  $f(m) = 2(-m) + 1$
- observe: since $n < m$
  $\Rightarrow -n > -m$
  $\Rightarrow 2(-n) > 2(-m)$
  $\Rightarrow 2(-n) + 1 > 2(-m) + 1$
  i.e. $f(n) > f(m)$.

hence $f(n) \neq f(m)$ in this case as well.

Case 3: $n \leq 0 < m$

- then $f(n) = 2(-n) + 1$ is odd
  $f(m) = 2m$ is even
- hence $f(n) \neq f(m)$ in this case as well.

↳ hence in all cases $f(n) \neq f(m)$

↳ since n, m arbitrary f is injective

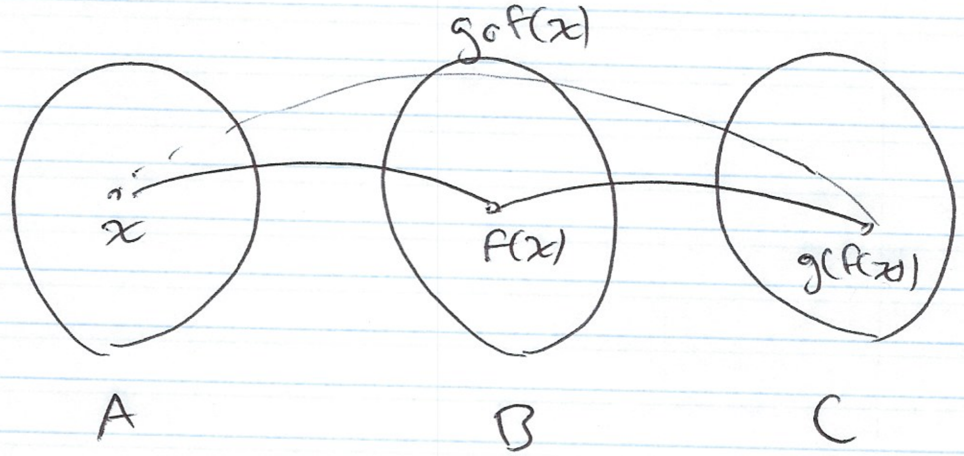
↳ hence f is bijective

Compositions

Def'n Sps $f: A \to B$ and $g: B \to C$ are functions.

The composition of f and g, denoted $g \circ f$, is defined by, $\forall x \in A$

$$g \circ f(x) = g(f(x))$$



ex: Define $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ by $f(m,n) = m+n$

$g: \mathbb{Z} \to \mathbb{N}$ by $g(n) = n^2+1$

Then $g \circ f(1,3) = g(f(1,3))$
$= g(4)$
$= 17$

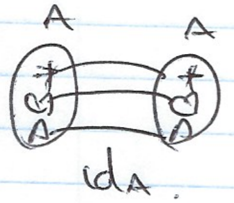
In general: $g \circ f(m,n)$
$= g(f(m,n))$
$= g(m+n)$
$= (m+n)^2 + 1$

The identity function

Def'n Let A be a fixed set. The identity function on A, denoted id_A, is the function defined by:

$$id_A : A \to A$$
$$\forall x \in A \quad id_A(x) = x.$$

e.g. if $A = \{*, \heartsuit, \Delta\}$
then $id_A : A \to A$ is:
$id_A = \{(*,*), (\heartsuit,\heartsuit), (\Delta,\Delta)\}$



Def'n Let $f : A \to B$ be a function. Then f is invertible iff there exists a function $g : B \to A$ s.t.

$$g \circ f = id_A \quad \text{i.e.} \quad \forall x \in A \; g(f(x)) = x$$
$$f \circ g = id_B \qquad\qquad \forall y \in B \; f(g(y)) = y.$$

g is called the inverse of f, and is denoted f^{-1}.

Note: not all functions are invertible!

In fact:

Theorem: Let $f: A \to B$ be a function. Then f is invertible iff f is a bijection.

($\Rightarrow$) Sps f is invertible. Let g be f's inverse. We prove f is a bijection.

(surjectivity): Fix $y \in B$.

Let $x = g(y)$

Then $f(x) = f(g(y))$

$= f \circ g(y)$

$= y$, since g = inverse of f.

Since y arbitrary, f is surjective ✓

(injectivity) - Fix $x, y \in A$ and sps $f(x) = f(y)$

- then $g(f(x)) = g(f(y))$
- i.e. $x = y$, since g is inverse of f

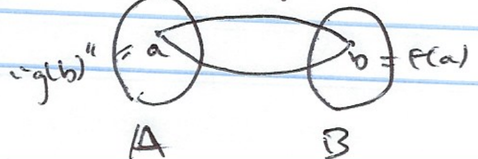
Since x, y arbitrary, f is injective ✓

↪ hence f is a bijection ✓

($\Leftarrow$) Sps f is a bijection from A to B. We prove f is invertible.

- Define $g = \{(b,a) \in B \times A \mid (a,b) \in f\}$

↪ we prove $g = f^{-1}$.



$(a,b) \in f \Rightarrow (b,a) \in g$

Claim 1: g is a function from B to A.

pf. WTS $\forall b \in B$ $\exists$ a unique $a \in A$ s.t. $(b,a) \in g$.

existence
- Fix $b \in B$. Since f is surjective $\exists a \in A$ $f(a) = b$ i.e. $(a,b) \in f$
- hence $(b,a) \in g$

uniqueness
- Suppose there is $a' \in A$ s.t. $(b, a') \in g$.
- Then must be (by def'n of g) that $(a', b) \in f$. i.e. $f(a') = b$.
- But then $f(a') = f(a)$
- hence since f is injective $a = a'$ ✓

Claim 2: $g = f^{-1}$

pf: - Fix $a \in A$.
- Let $b = f(a)$, so that $(a,b) \in f$
- then $(b,a) \in g$, i.e. $g(b) = a$
- hence $g(f(a)) = g(b) = a$
- Since a arbitrary, $g \circ f = id_A$. ✓

- Fix $b \in B$.
- Let $a = g(b)$, i.e. $(b,a) \in g$.
- Then $(a,b) \in f$ by def'n of g i.e. $f(a) = b$
- hence $f(g(b)) = f \circ g(b) = b$
- Since b arbitrary, $f \circ g = id_B$ ✓

hence g is inverse of f ✓

→ Can use theorem to prove certain functions are bijections

<u>Ex:</u> Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x - 5$

<u>Claim</u> f is a bijection

<u>Pf:</u> - we show f is invertible.
- Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = (x+5)/2$
- we show $g = f^{-1}$

<u>Now:</u> - fix $x \in \mathbb{R}$.
- $g \circ f(x) = g(f(x))$
$= g(2x-5)$
$= ((2x-5)+5)/2$
$= x$

$\Rightarrow g \circ f = id_{\mathbb{R}}$.

<u>also:</u>
- $f \circ g(x) = f(g(x))$
$= f\left(\frac{x+5}{2}\right)$
$= 2\left(\frac{x+5}{2}\right) - 5$
$= x$

$\Rightarrow f \circ g = id_{\mathbb{R}}$

hence f is invertible
hence f is a bijection, by previous theorem