Partitions yield equiv. relations

idea: if $\mathbb{P}$ is a partition on A, can define equiv. relation R on A by rule "$(x,y) \in R$ iff x and y are in same piece of partition"

Picture

$(x,y) \in R$ but $(x,z) \notin R$

Let's prove this works:

Theorem Sps $\mathbb{P}$ is a partition of A. Define a relation $R_{\mathbb{P}}$ on A by:

$$(x,y) \in R_{\mathbb{P}} \text{ iff } \exists X \in \mathbb{P} \text{ s.t. } x \in X \text{ and } y \in X.$$

Then $R_{\mathbb{P}}$ is an equivalence relation.

Pf: (i) reflexivity:
- Fix ~~xxxx~~. $x \in A$.
- Since $\mathbb{P}$ is a partition of A, there is $X \in \mathbb{P}$ s.t. $x \in X$

using $\bigcup_{X \in \mathbb{P}} X = A$

- hence $x \in X$ also.
- hence $(x,x) \in R_{\mathbb{P}}$ ✓

(ii) <u>symmetry</u>
- Fix $x, y \in A$ and suppose $(x,y) \in R_{\mathbb{P}}$
- Then there is $X \in \mathbb{P}$ s.t. $x \in X$ <u>and</u> $y \in X$
- Hence $y \in X$ and $x \in X$
- hence $(y,x) \in R_{\mathbb{P}}$ ✓

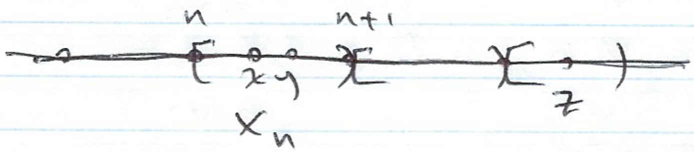
(iii) <u>transitivity</u>
- Fix $x, y, z \in A$ and sps $(x,y) \in R_{\mathbb{P}}$ and $(y,z) \in R_{\mathbb{P}}$
- then $\exists X \in \mathbb{P}$ s.t. $x \in X$ and $y \in X$ and $\exists Y \in \mathbb{P}$ s.t. $y \in Y$ and $z \in Y$
- hence $y \in X \cap Y$
- in particular $X \cap Y \neq \emptyset$, so that $X = Y$ (since $\mathbb{P}$ is a partition)
- hence $x \in X$ and $z \in X$ i.e. $(x,z) \in R_{\mathbb{P}}$ ✓

<u>Ex's</u> ① - Let $\mathbb{P} = \{X_n : n \in \mathbb{Z}\}$ be our partition of $\mathbb{R}$ from last time, i.e. $X_n = [n, n+1)$
- Let $R_{\mathbb{P}}$ be the associated equivalence relation: $(x,y) \in R_{\mathbb{P}}$ iff $\exists n$ s.t. $x \in X_n$ and $y \in X_n$

i.e. ~~[illegible]~~

$x \in [n, n+1)$ and $y \in [n, n+1)$
- by our theorem, this defines an equiv relation.
- easy to see this is the

same equiv. relation R that we defined previously in a different way: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$



$(x,y) \in R = R_{\mathbb{P}}$
$(x,z) \notin R = R_{\mathbb{P}}$

notice: the equivalence classes of this ~~equivalence~~ equiv. relation are exactly the pieces in the partition

(2) - let $\mathbb{P} = \{\{1\}, \{2,3,4\}\}$ be our partition of $A = \{1,2,3,4\}$ from last time.

- let $R_{\mathbb{P}}$ be the associated equiv. relation

- so e.g. $(1,1) \in R_{\mathbb{P}}$
$(2,3) \in R_{\mathbb{P}}$
but $(1,3) \notin R_{\mathbb{P}}$.

- In this case we can explicitly write out $R_{\mathbb{P}}$ as a set in roster notation:

$$R_{\mathbb{P}} = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}$$

– no real rhyme or reason to this equiv. relation, but still a perfectly good one.

Equiv. relations yield partitions

– summary of chars: given a partition $\mathbb{P}$ of A, can define an equiv. relation $R_{\mathbb{P}}$ by saying the equivalence classes of $R_{\mathbb{P}}$ are exactly the pieces of the partition $\mathbb{P}$.

– Conversely: given an equiv. relation R on A, (you will prove on HW!) the equiv. classes of R always form a partition of A.

Def'n Sps R is an equiv. relation on A. We denote the set of equiv classes of R as A/R

i.e. $A/R = \{ [x]_R : x \in A \}$

"A mod R"

Ex's – Consider $\equiv_3$ on $\mathbb{Z}$.

– Then:

$$\mathbb{Z}/\equiv_3 = \{ [n]_3 : n \in \mathbb{Z} \}$$

$$= \{\ldots, [-1]_3, [0]_3, [1]_3, [2]_3, \ldots\}$$

we checked already:

$$\ldots [-3]_3 = [0]_3 = [3]_3 = [6]_3 \ldots$$
$$\ldots = [1]_3 = [4]_3 = [7]_3 \ldots$$
$$\ldots = [2]_3 = [5]_3 = [8]_3 \ldots$$

so really:

"set of remainders" $\mathbb{Z}/\equiv_3 = \{[0]_3, [1]_3, [2]_3\}$

we could as well write:

$$\mathbb{Z}/\equiv_3 = \{[3]_3, [4]_3, [5]_3\}.$$

<u>NOTATION</u>: it is conventional to write $\mathbb{Z}/\equiv_n$ as $\mathbb{Z}/n\mathbb{Z}$

"$\mathbb{Z}$ mod $n\mathbb{Z}$"

just like with 3, in general we have:

$$\mathbb{Z}/n\mathbb{Z} = \{[0]_n, [1]_n, \ldots, [n-1]_n\}.$$

② - let R be the "floor" equiv. relation on $\mathbb{R}$: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$

- we knew from before: equiv. classes are intervals of the form $[n, n+1)$

- if $x \in [n, n+1)$ then $[x]_R = [n, n+1)$

- <u>any</u> x in this interval serves equally well as a representative of the equiv. class

- so e.g. $[0]_R = [1/2]_R = [3/5]_R$

$$= [0,1)$$

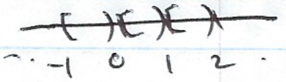
$$[1]_R = [1.2121\ldots]_R = [1.99]_R$$

$$= [1,2)$$

etc.

we have:

$$\mathbb{R}/R = \{ [x]_R : x \in \mathbb{R} \}$$



$$= \{ \ldots, [-1,0), [0,1), [1,2), \ldots \}$$

$$= \{ \ldots, [-1]_R, [0]_R, [1]_R, \ldots \}$$

$$= \{ \ldots, [-1/2]_R, [1/2]_R, [3/2]_R, \ldots \}$$

etc.

- In both ex's 1 and 2 the set of equiv classes forms a partition
- turns out this is always the case.

Thm If R is an equiv. relation on A, then A/R is a partition of A.

Pf: HW. For hint, see problem 6.7.13 pg. 449, which outlines the proof.

Order Relations

→ Another common type of binary relation is an order relation
→ come in several flavors:
  nonstrict / strict
and partial / total.

Def'n - A relation R on a set A is a (nonstrict) partial order iff R is reflexive, transitive, and antisymmetric.

- if R is a partial order on A we say that the pair (A, R) is a partially ordered set or poset

(i)

Ex's 1) $\leq$ is a partial order on $\mathbb{R}$.

pf: $\forall x, y, z \in \mathbb{R}$ we have:

$x \leq x$ ✓

if $x \leq y$ and $y \leq x$ then $x = y$ ✓

if $x \leq y$ and $y \leq z$ then $x \leq z$ ✓

$\hookrightarrow$ so $(\mathbb{R}, \leq)$ is a poset.

2) Let A be any set. Then the subset relation $\subseteq$ on $P(A)$ is a partial order

pf: $\forall X, Y, Z \in P(A)$ we have:

$X \subseteq X$ ✓.

if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$ ✓

if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$ ✓

$\hookrightarrow$ so $(P(A), \subseteq)$ is a poset.

3) - We showed before that the divisibility relation on $\mathbb{N}$ is refl., trans., antisymmetric, hence $(\mathbb{N}, |)$ is a poset

- We also showed $|$ is not antisymmetric on $\mathbb{Z}$. Hence $(\mathbb{Z}, |)$ is not a poset.

$\hookrightarrow$ these ex's seem to be of different kinds: yet: any theorems that can be proved using only properties of reflexivity, transitivity and antisymmetry must hold for all three! (and any other poset).

Strict p.o.'s

Def'n a relation R on A is called irreflexive iff $(\forall x \in A)((x,x) \notin R)$

↳ e.g. $<$ and $\neq$ are irreflexive since we never have $x < x$ or $x \neq x$.

Def'n a relation R on a set A is called a strict partial order iff R is irreflexive, transitive, and antisymmetric.

↳ this is official def'n, but by HW this is equiv to being transitive and asymmetric $\forall x, y$ $xRy \Rightarrow y \not R x$

Ex's ① $<$ is a strict partial order on $\mathbb{R}$.

pf: $\forall x, y, z \in \mathbb{R}$ we have:

(i) $x \not< x$ ✓

(ii) $x < y \wedge y < z \Rightarrow x < z$ ✓

always false

(iii) $x < y \wedge y < x \Rightarrow x = y$ ✓

(iii)

by HW: could combine (i) and (iii) by observing:

(iv) $x < y \Rightarrow y \not< x$ ✓

② $\subsetneq$ is a strict partial order on $P(A)$, for any set A.

pf: $\forall X, Y, Z \in P(A)$ we have:

$$X \subsetneq Y \wedge Y \subsetneq Z \Rightarrow X \subsetneq Z \checkmark$$
$$X \subsetneq Y \Rightarrow \neg (Y \subsetneq X) \checkmark$$

<u>nonexamples</u>

① $\leq$ is <u>not</u> a strict partial order on $\mathbb{R}$ since it is <u>not</u> irreflexive (in fact $\leq$ is reflexive: this is stronger than being not irreflexive!) Sim'ly $\subseteq$ is <u>not</u> a strict partial order on any set.

② $<$, $\subsetneq$ are not (nonstrict) partial orders: neither are reflexive.

③ $\neq$ (e.g. on $\mathbb{N}$) is neither a partial order nor strict partial order since transitivity fails: e.g. $2 \neq 5$ and $5 \neq 2$ but $2 = 2$ ✓

Total orders

Def'n a relation R on A is said to be total iff
$$(\forall x, y \in A)\ ((x,y) \in R \vee (y,x) \in R \vee x = y)$$

Def'n - if R is a partial order on A that is also total, then R is called a total order on A.
- if R is a strict partial order on A that is also total then R is called a strict total order on A

Ex's ① $\leq$ is a total order on $\mathbb{R}$:
We know already that $\leq$ is a partial order and:
$$(\forall x, y \in \mathbb{R})\ x \leq y \vee y \leq x \vee x = y \checkmark$$

② $\subseteq$ is not a total order on $P(\mathbb{N})$.

e.g. If $X = \{1, 2, 3\}$
$Y = \{3, 4\}$

then $X \not\subseteq Y$
$Y \not\subseteq X$
and $X \neq Y$

③ $<$ is a strict total order on $\mathbb{R}$ since $(\forall x, y \in \mathbb{R})\ x < y \vee y < x \vee x = y$

(v)

Ex

a strict partial order on $\mathbb{N}\times\mathbb{N}$:

Define a relation R on $\mathbb{N}\times\mathbb{N}$ by:

$$(n_1, m_1)\ R\ (n_2, m_2) \text{ iff } n_1 < n_2 \text{ and } m_1 < m_2$$

So e.g. $(1,2)\ R\ (3,5)$ since $1<3$ and $2<5$

but $(3,1)\ \not R\ (2,2)$ since $3 \not< 2$.

Claim: R is a strict partial order on $\mathbb{N}\times\mathbb{N}$

Pf: we prove (1) transitivity
(2) asymmetry

(1) Fix $(n_1, m_1), (n_2, m_2), (n_3, m_3) \in \mathbb{N}\times\mathbb{N}$ and suppose

$(n_1, m_1)\ R\ (n_2, m_2)$

and $(n_2, m_2)\ R\ (n_3, m_3)$

then: $n_1 < n_2$ and $m_1 < m_2$
and $n_2 < n_3$ and $m_2 < m_3$

hence by transitivity of $<$

(vi) (30)

$n_1 < n_3$ and $m_1 < m_3$

hence $(n_1, m_1)\ R\ (n_3, m_3)$ ✓

(2) Fix $(n_1, m_1), (n_2, m_2) \in \mathbb{N} \times \mathbb{N}$

Sps $(n_1, m_1)\ R\ (n_2, m_2)$
then

$n_1 < n_2$ and $m_1 < m_2$

hence $n_2 \not< n_1$ (and $m_2 \not< m_1$)

hence $(n_2, m_2) \not R (n_1, m_1)$ ✓

<u>Observe:</u> R defined above is <u>not</u> total: e.g.

$(1,2) \not R (3,1)$
and $(3,1) \not R (1,2)$
and $(1,2) \neq (3,1)$