Binary Relations

- Binary relations are ubiquitous in math
- e.g. we have order relations like $x \leq y$

  $x < y$

  the subset relation $X \subseteq Y$

  the divisibility relation $n | m$ ("n divides m")

→ all assert a relation between two mathematical objects (hence "binary")

- What are $\leq$, $<$, $\subseteq$, $|$, etc. as mathematical objects themselves?

- We will define binary relations as sets of ordered pairs

Def'n - Sps A, B are sets. A binary relation on A and B is just a subset $R \subseteq A \times B$

- if $(a,b) \in R$ we say "a is related to b" and sometimes write $a R b$.

- A is the domain of R; B is the codomain
- often we have $A = B$, so that $R \subseteq A \times A$. In this case we say: R is a relation on A

<u>Ex's</u> - ① Let A = set of Shakespeare's characters, B = set of Shakespeare's plays

- Define a relation $R \subseteq A \times B$ by:

$(a,b) \in R$ iff a appears in ~~a~~ b

- then:

(Romeo, "Romeo and Juliet") $\in R$

(Iago, "Othello") $\in R$

but (Romeo, "Othello") $\notin R$

- could also write:

Romeo R "Romeo and Juliet"

Iago R "Othello"

Romeo $\not R$ "Othello"

to express this

② Consider the relations $\leq$, $<$ on $\mathbb{N}$: can think of them as sets of pairs:

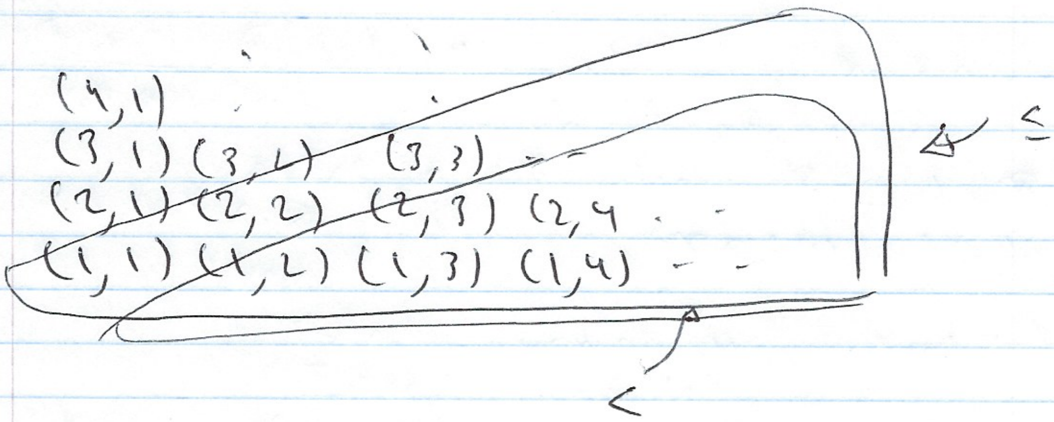
$$\leq \; = \{(1,1), (1,2), (1,3), (2,3), (3,1000)\}$$

$$< \; = \{(1,2), (1,3), (2,3), \ldots\}$$

- Instead of writing $(1,2) \in <$ we usually write $1<2$, but these assert the same thing

- likewise $2 \not< 1$ asserts $(2,1) \notin <$

- if we visualize $\mathbb{N} \times \mathbb{N}$ as a grid then $\leq$ and $<$ are "lower triangular half-grids"



3. Let A be a set. We can think of $=$ as a binary relation on A:

$=$ is the set $\{(x,x) : x \in A\}$

<u>Properties</u> <u>relations</u> <u>can have</u>:

<u>Def'n</u> Spse A is a set and $R \subseteq A \times A$ is a relation on A

1. R is <u>reflexive</u> iff $(\forall x \in A)((x,x) \in R)$
2. R is <u>symmetric</u> iff $(\forall x,y \in A)((x,y) \in R \Rightarrow (y,x) \in R)$
3. R is <u>transitive</u> iff $(\forall x,y,z \in A)((x,y) \in R \wedge (y,z) \in R \Rightarrow (x,z) \in R)$
4. R is <u>antisymmetric</u> iff $(\forall x,y \in A)((x,y) \in R \wedge (y,x) \in R \Rightarrow x = y)$

Ex's ① on any set A, the equality relation $=$ is always reflexive, symmetric, and transitive

(also anti-symmetric ...)

↳ relations w/ these 3 properties are called equivalence relations

② $\leq$ (e.g. on $\mathbb{N}$) is reflexive, transitive, and antisymmetric:
why: $(\forall n \in \mathbb{N})\ (n \leq n)$ ✓
$(\forall n, m, \ell \in \mathbb{N})(n \leq m \wedge m \leq \ell \Rightarrow n \leq \ell)$ ✓
$(\forall n, m \in \mathbb{N})\ (n \leq m \wedge m \leq n \Rightarrow n = m)$ ✓

but $\leq$ is not symmetric: e.g. $3 \leq 5$ but $5 \not\leq 3$.

③ $<$ (e.g. on $\mathbb{N}$) is not reflexive, symmetric, but is transitive
(Q: is $<$ antisymmetric?)

④ Let $A = \{$rock, paper, scissors$\}$
Define a relation R on A by $(a,b) \in R$ iff a beats b.
Then R is not transitive, since:
(scissors, paper) $\in R$
and (paper, rock) $\in R$
but (scissors, rock) $\notin R$.

5. Consider the divisibility relation $|$ on $\mathbb{N}$:

$n|m$ iff n divides m

i.e. iff $(\exists k \in \mathbb{N})(m = nk)$

e.g. $2|4$ and $2|6$ but $2 \nmid 9$.

<u>Claim</u>: the divisibility relation $|$ on $\mathbb{N}$ is:

(i) reflexive
(ii) <u>not</u> symmetric
(iii) transitive
(iv) antisymmetric

Pf: (i) For any $n \in \mathbb{N}$ we have $n|n$ since $n = n \cdot 1$

(ii) $2|4$ but $4 \nmid 2$

(iii) Suppose $n, m, l \in \mathbb{N}$ and $n|m$ and $m|l$, i.e.

$$m = nk_1$$
$$l = mk_2$$

then
$$l = (nk_1)k_2 = n(k_1 k_2)$$

hence $n|l$ ✓

(iv) Suppose $n, m \in \mathbb{N}$ and $n|m$ and $m|n$

then
$$n = k_1 m$$
$$m = k_2 n$$

so that
$$n = k_1 k_2 n$$

$\Rightarrow k_1 k_2 = 1 \Rightarrow k_1 = k_2 = 1$

$\Rightarrow n = m$ ✓

6 now consider | on $\mathbb{Z}$, defined by $n|m$ iff

$$(\exists k \in \mathbb{Z})(m = nk)$$

Then | is still reflexive, transitive but no longer antisymmetric:

$2 | -2$

$-2 | 2$ but $2 \neq -2$. ✓

Equivalence Relations

Relations satisfying properties 1, 2, 3 have a special name:

Def'n A relation R on a set A is called an *equivalence relation* iff R is reflexive, symmetric, and transitive.

Ex's 1 Let A be a set and consider the equality relation $=$ on A.

Then $=$ is an equivalence relation.

Pf. $\forall x, y, z \in A$ we have:

$x = x$ ✓

$x = y \Rightarrow y = x$ ✓

$x = y \wedge y = z \Rightarrow x = z$ ✓

2. Recall: the floor of a real number x, denoted $\lfloor x \rfloor$, is the unique integer n s.t. $n \le x < x+1$.

e.g. $\lfloor 1.5 \rfloor = 1$
$\lfloor \pi \rfloor = 3$
$\lfloor -2.739 \rfloor = -3$
$\lfloor 5 \rfloor = 5$

Define a relation R on $\mathbb{R}$ by: $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$
equivalently
$$R = \{(x,y) \in \mathbb{R}^2 \mid \lfloor x \rfloor = \lfloor y \rfloor\}.$$

__Claim__ R is an equivalence relation.

__Pf__. (i) Fix $x \in \mathbb{R}$. Then $\lfloor x \rfloor = \lfloor x \rfloor$. ✓ Hence $(x,x) \in R$. Since x was arbitrary, we have $(\forall x \in \mathbb{R})((x,x) \in R)$

(ii) Fix $x, y \in \mathbb{R}$. If $\lfloor x \rfloor = \lfloor y \rfloor$ ✓ then $\lfloor y \rfloor = \lfloor x \rfloor$. That is, if $(x,y) \in R$ then $(y,x) \in R$.

(iii) Fix $x, y, z \in \mathbb{R}$. If $\lfloor x \rfloor = \lfloor y \rfloor$ and $\lfloor y \rfloor = \lfloor z \rfloor$ then $\lfloor x \rfloor = \lfloor z \rfloor$. ✓ That is, if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

3. More generally suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function.

Define a relation R_f by:
$(x,y) \in R_f$ iff $f(x) = f(y)$
i.e.
$R_f = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid f(x)=f(y)\}$

then R_f is an equivalence relation

Pf: homework.

(4) Define a relation $\equiv_3$ on $\mathbb{Z}$ as follows:
$(n,m) \in \equiv_3$ iff $3 \mid (m-n)$

i.e. $\equiv_3 = \{(n,m) \in \mathbb{Z}^2 \mid 3 \mid (m-n)\}$

↳ we'll write $n \equiv_3 m$ for $(n,m) \in \equiv_3$.

e.g. $2 \equiv_3 5$ since $3 \mid (5-2)$
$7 \equiv_3 -2$ since $3 \mid (-2-7)$
$6 \not\equiv_3 7$ since $3 \nmid (7-6)$

$n \mid m$ means $\exists k$ $m = nk$

Claim: $\equiv_3$ is an equivalence relation on $\mathbb{Z}$.

Pf. (i) Fix $n \in \mathbb{Z}$. Observe that $3 \mid (n-n)$, i.e. $3 \mid 0$, since $0 = 3\cdot 0$
Thus ~~[illegible]~~ $n \equiv_3 n$ ✓

So $n \mid 0$ since $0 = n\cdot 0$

(ii) Fix $n, m \in \mathbb{Z}$ and suppose $n \equiv_3 m$. We prove $m \equiv_3 n$.

pf: since $n \equiv_3 m$ we have $3 \mid (m-n)$, i.e. $(\exists k \in \mathbb{Z})(m-n = 3k)$

But then $n-m = 3(-k)$

hence $3 \mid n-m$

i.e. $m \equiv_3 n$ ✓

(iii) Fix $n, m, \ell \in \mathbb{Z}$ and suppose $n \equiv_3 m$ and $m \equiv_3 \ell$. We prove $n \equiv_3 \ell$.

pf: $\exists k_1, k_2 \in \mathbb{Z}$ s.t.

$$m - n = 3k_1$$
$$\ell - m = 3k_2$$

adding these equations gives:

$$(m-n) + (\ell - m) = 3k_1 + 3k_2$$

i.e. $\ell - n = 3(k_1 + k_2)$

hence $3 \mid \ell - n$

i.e. $n \equiv_3 \ell$ ✓

→ $\equiv_3$ is called <u>equivalence modulo 3</u>

→ it is more common to write $n \equiv_3 m$ as $n \equiv m \pmod 3$

the way to think about it:

$n \equiv_3 m$ if and only if n and m have same remainder when divided by 3.

(10)

e.g $2 \equiv 5 \pmod 3$
since $2 = 3 \cdot 0 + 2$ ← same
$5 = 3 \cdot 1 + 2$ ← remainder

$7 \equiv 13 \pmod 3$
since $7 = 3 \cdot 2 + 1$
$13 = 3 \cdot 4 + 1$

$7 \equiv -2 \pmod 3$
since $7 = 3 \cdot 2 + 1$
$-2 = 3(-1) + 1$

$7 \not\equiv 11 \pmod 3$
since $7 = 3 \cdot 2 + 1$ ← diff
$11 = 3 \cdot 3 + 2$ ← remainders

(3) There is nothing special about 3. For any fixed $k \in \mathbb{N}$ can define $\equiv_k$ on $\mathbb{Z}$ by:

$n \equiv_k m$ iff $k \mid (m-n)$
(iff m, n have same remainder when divided by k)

↳ again we usually write
$n \equiv m \pmod k$
for
$n \equiv_k m$.

↳ all of these "congruence modulo k" relations are equivalence relations ✓

Nonexamples

1. Consider $\leq$ (e.g. on $\mathbb{R}$)
reflexive ✓, transitive ✓,
but not symmetric,
hence not an equiv relation

2. Consider the inequality relation $\neq$ on $\mathbb{Z}$
symmetric since $n \neq m \Rightarrow m \neq n$
but not reflexive (in fact never true that $n \neq n$)
nor transitive (e.g. $2 \neq 4$ and $4 \neq 2$ but $2 = 2$)

Equivalence classes

- Suppose R is an equivalence relation on a set A.

Defn For each fixed $x \in A$, the equivalence class of x, denoted $[x]_R$, is the set of el'ts related to x by R:

$$[x]_R = \{ y \in A \mid (x,y) \in R \}$$

by symmetry could have defined as

$\{y \in A \mid (y,x) \in R\}$

- Warning: overloaded notation
- we used [ ]'s when writing $[n] = \{1, 2, \ldots, n\}$

↪ this is completely unrelated meaning to $[x]_R$ for an equiv relation R.
— Beware! Don't get confused.

<u>Ex's</u> ① Let $=$ be the equality relation on $\mathbb{Z}$. Then for any fixed $n \in \mathbb{Z}$, we have

$$[n]_= \overset{\text{def}}{=} \{ m \in \mathbb{Z} \mid n = m \}$$

$$= \{n\}$$

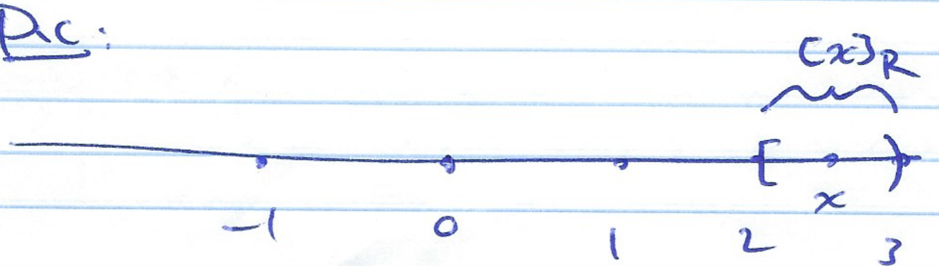
② Let R denote the floor equiv. relation on $\mathbb{R}$, i.e. $(x,y) \in R$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.

Now: Fix $x \in \mathbb{R}$ and suppose $\lfloor x \rfloor = n$.

<u>Then</u>:

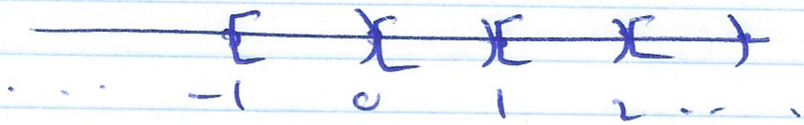
$$\begin{aligned} [x]_R &= \{ y \in \mathbb{R} \mid (x,y) \in R \} \\ &= \{ y \in \mathbb{R} \mid \lfloor x \rfloor = \lfloor y \rfloor \} \\ &= \{ y \in \mathbb{R} \mid n = \lfloor y \rfloor \} \\ &= \{ y \in \mathbb{R} \mid n \le y < n+1 \} \\ &= [n, n+1) \end{aligned}$$

<u>Pic</u>:



e.g. $[\pi]_R = [3,4)$

<u>notice</u>: the equivalence classes of R <u>partition</u> $\mathbb{R}$!



we'll prove later this always happens.

(3) Consider $\equiv_3$, equivalence modulo 3 on $\mathbb{Z}$: $m \equiv_3 n$ iff $3 \mid n-m$.

What are the equiv. classes of this relation?

Let's write some down.

$$
\begin{aligned}
[0]_{\equiv_3} &= \{n \in \mathbb{Z} \mid 0 \equiv_3 n\} \\
&= \{n \in \mathbb{Z} \mid 3 \mid n-0\} \\
&= \{n \in \mathbb{Z} \mid 3 \mid n\} \\
&= \{\ldots -3, 0, 3, 6, \ldots\}
\end{aligned}
$$

$$
\begin{aligned}
[1]_{\equiv_3} &= \{n \in \mathbb{Z} \mid 1 \equiv_3 n\} \\
&= \{n \in \mathbb{Z} \mid 3 \mid n-1\} \\
&= \{n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\, n = 3k+1\} \\
&= \{\ldots -5, -2, 1, 4, 7, \ldots\}
\end{aligned}
$$

$$[2]_{\equiv_3} = \{ n \in \mathbb{Z} \mid 2 \equiv_3 n \}$$
$$= \{ n \in \mathbb{Z} \mid 3 \mid n-2 \}$$
$$= \{ n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})\ n = 3k+2 \}$$
$$= \{ \ldots -4, -1, 2, 5, \ldots \}$$

$$[3]_{\equiv_3} = \{ n \in \mathbb{Z} \mid 3 \equiv_3 n \}$$
$$= \{ n \in \mathbb{Z} \mid 3 \mid n-3 \}$$
$$= \{ n \in \mathbb{Z} \mid (\exists k \in \mathbb{Z})(n = 3k+3) \}$$
$$= \{ n \in \mathbb{Z} \mid (\exists \ell \in \mathbb{Z})(n = 3\ell) \}$$
$$= \{ \ldots -3, 0, 3, 6, \ldots \}$$

$$= [0]_{\equiv_3}$$

and can see

$$[4]_{\equiv_3} = [1]_{\equiv_3}$$
$$[5]_{\equiv_3} = [2]_{\equiv_3}$$
$$[6]_{\equiv_3} = [3]_{\equiv_3} = [0]_{\equiv_3} \quad \text{etc...}$$

→ equiv. classes consist of all $n \in \mathbb{Z}$ of a given remainder when divided by 3

→ again we see: the equiv. classes form a <u>partition</u> of $\mathbb{Z}$.

$$\mathbb{Z} = \{\ldots -3, 0, 3, 6, \ldots\} \cup \{\ldots -5, -2, 1, 4, 7, \ldots\} \cup \{\ldots -4, -1, 2, 5, \ldots\}$$

pairwise disjoint

$$= [0]_{\equiv_3} \cup [1]_{\equiv_3} \cup [2]_{\equiv_3}$$

Notation: for equivalence mod n we usually write $[x]_n$ instead of $[x]_{\equiv_n}$.

e.g. we'll write

$$\mathbb{Z} = [0]_3 \cup [1]_3 \cup [2]_3 .$$

Our next goal is to see that "partition" and "equivalence relation" are in fact two names for the same concept.

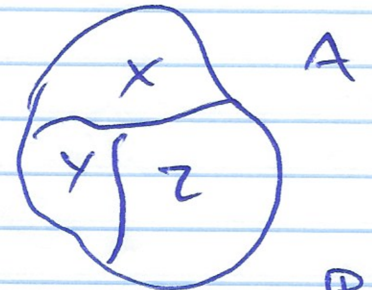
Recall: if A is a set, a partition $\mathbb{P}$ of A is a collection of subsets of A (i.e. $\mathbb{P} \subseteq P(A)$) s.t.

1. $(\forall X \in \mathbb{P})\ X \neq \emptyset$.
2. $(\forall X, Y \in \mathbb{P})\ X \neq Y \Rightarrow X \cap Y = \emptyset$
3. $\bigcup_{X \in \mathbb{P}} X = A$.

Note: before we indexed our partitions .. irrelevant note: - ② says the pieces of the partition are pairwise disjoint

- can also write in equiv form: $(\forall X, Y \in \mathbb{P})\ (X = Y \text{ or } X \cap Y = \emptyset)$

Pic:



$\mathbb{P} = \{X, Y, Z\}$
a partition of A.

ex's ① let

$A = \{\ldots, -3, 0, 3, 6, \ldots\}$
$B = \{\ldots, -2, 1, 4, 7, \ldots\}$
$C = \{\ldots, -1, 2, 5, 8, \ldots\}$

then $\mathbb{P} = \{A, B, C\}$ is a partition of $\mathbb{Z}$

pf: ① $A, B, C \neq \emptyset$ ✓

② $A \cap B = A \cap C = B \cap C = \emptyset$ ✓

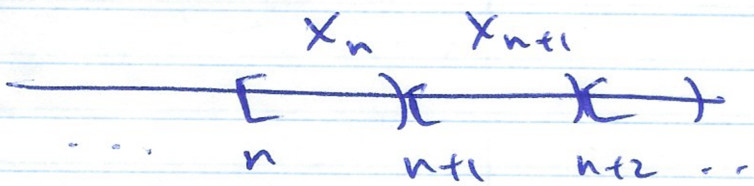
③ $\bigcup_{X \in \mathbb{P}} X = A \cup B \cup C = \mathbb{Z}$ ✓

2 For every $n \in \mathbb{Z}$, define

$$X_n = \{x \in \mathbb{R} \mid n \leq x < n+1\}$$
$$= [n, n+1)$$

Then $P = \{X_n : n \in \mathbb{Z}\}$

is a partition of $\mathbb{R}$.



Pf: you try.

3 Let $A = \{1, 2, 3, 4\}$

then if $X = \{1\}$
$Y = \{2, 3, 4\}$

then $P = \{X, Y\}$
$= \{\{1\}, \{2, 3, 4\}\}$

is a partition of A.