Note: prop'n is one version of "AMGM" Inequality

↳ arithmetic mean (AM) of x, y is $x+y/2$

↳ geometric mean (GM) of $x, y \geq 0$ is $\sqrt{xy}$

prop'n gives for $x, y \geq 0$

$$\sqrt{xy} \leq x+y/2$$

i.e. $GM \leq AM$.

Indirect Strat:
- Assume $\neg(\forall x \in S)\, P(x)$ (i.e. $(\exists x \in S)\, \neg P(x)$) and get a contradiction

Ex ② Prop'n $\sqrt{2}$ is irrational, that is,

$$(\forall a, b \in \mathbb{Z}) \left(\frac{a}{b} \neq \sqrt{2}\right)$$

Pf: - Suppose not, that is, suppose $\exists a, b \in \mathbb{Z}$ s.t.

$$a/b = \sqrt{2}$$

- we may assume a/b is in reduced form, i.e. a and b have no common factors since if they do we can cancel and get a fraction in reduced form.

Now since
$$\frac{a}{b} = \sqrt{2}$$
we have
$$a = \sqrt{2}\, b$$
hence
$$a^2 = 2b^2$$

Hence a^2 is even. It follows a is even (why?)

Hence $\exists k \in \mathbb{Z}$ s.t. $a = 2k$.

So then $a^2 = 4k^2$

But then $2b^2 = 4k^2$
so that $b^2 = 2k^2$

The same reasoning shows b is even too.

But then <u>both</u> a, b are even and therefore share a factor of 2.

A contradiction, as we assumed a, b shared no common factors

The prop'n follows!

Conditional Claims

General Form: $P \Rightarrow Q$

Three Strategies: ① Direct: Assume P holds, prove Q.

② Contrapositive: Show $\neg Q \Rightarrow \neg P$

i.e. assume $\neg Q$ and prove $\neg P$.

③ Indirect: Assume $\neg(P \Rightarrow Q)$,

i.e. assume $P \wedge \neg Q$; derive a contradiction

really the same

Ex's ① (Direct) Let $O = \{\ldots -3, -1, 1, 3, 5, \ldots\}$ denote the set of odd integers (including negatives)

Prop'n $(\forall n \in \mathbb{Z})(n \in O \Rightarrow n^2 - 1$ is divisible by 4)

or, even more symbolically

$$(\forall n \in \mathbb{Z})(n \in O \Rightarrow (\exists k \in \mathbb{Z})(n^2 - 1 = 4k))$$

Pf: overall, a universal claim, so begin as usual:

- Fix $n \in \mathbb{Z}$ arbitrary
- now we deal w/ conditional: assume $n \in O$.
- hence $\exists k \in \mathbb{Z}$ s.t. $n = 2k + 1$
- hence

$$n^2 = (2k+1)^2 = 4k^2 + 4k - 1$$

hence $n^2 - 1 = 4k^2 + 4k$
$= 4(k^2 + 4)$ ✓

hence $n^2 - 1$ is divisible by 4 ✓

since n was arbitrary, done ✓

2 (Contrapositive) Let E be set of even integers (not necessarily positive)

Prop'n $(\forall m, n \in \mathbb{Z})$ (if mn is even, then either m or n is even)

symbolically: $(\forall m, n \in \mathbb{Z})\ (mn \in E \Rightarrow [(m \in E) \vee (n \in E)])$

Pf: - Fix $m, n \in \mathbb{Z}$ arbitrary
- Suppose $\neg[m \in E \vee n \in E]$
i.e. $m \notin E \wedge n \notin E$

- then m and n are odd.
- hence $\exists k, \ell \in \mathbb{Z}$ s.t.
$m = 2k+1$
$n = 2\ell+1$

- hence
$$\begin{aligned} mn &= (2k+1)(2\ell+1) \\ &= 4k\ell + 2k + 2\ell + 1 \\ &= 2(2k\ell + k + \ell) + 1 \\ &= 2M + 1 \end{aligned}$$

where $M = 2k\ell + k + \ell$

- hence mn is odd, i.e. $mn \notin E$

- we've proved $m \notin E \wedge n \notin E \Rightarrow mn \notin E$

i.e. $\neg(m \in E \vee n \in E) \Rightarrow \neg(mn \in E)$

- by contrapositive we have:

$$mn \notin E \Rightarrow m \notin E \vee n \notin E$$

- since m, n were arbitrary, done ✓

(indirect) 8) <u>Prop'n</u> $(\forall x \in \mathbb{R})(x > 0 \Rightarrow x + 1/x \geq 2)$

<u>Pf</u>: - Fix $x \in \mathbb{R}$.
- Suppose $\underset{P}{x > 0}$ but $\underset{\neg Q}{x + 1/x < 2}$

$$\Rightarrow x^2 + 1 < 2x$$
$$\Rightarrow x^2 - 2x + 1 < 0$$
$$\Rightarrow (x-1)^2 < 0$$

a contradiction as $(x-1)^2 \geq 0$ as it is a square.

Hence we must have

$$x > 0 \Rightarrow x + 1/x \geq 2$$

Since x was arbitrary, we are done ✓

Biconditional Claims

General Form: $P \Leftrightarrow Q$

Strategy: Prove $P \Rightarrow Q$ and
$Q \Rightarrow P$

Ex Propn An integer is even if and only if its square is even.

i.e.
$$(\forall n \in \mathbb{Z})\ (n \in E \Leftrightarrow n^2 \in E)$$

Pf: Fix $n \in \mathbb{Z}$ arbitrary

($\Rightarrow$) - Suppose $n \in E$.
- Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$
- hence
$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \\ &= 2M \quad \text{where } M = 2k^2 \end{aligned}$$
- hence n^2 is even ✓

($\Leftarrow$) - To prove $n^2 \in E \Rightarrow n \in E$
we prove the contrapositive
i.e.
$$n \notin E \Rightarrow n^2 \notin E$$

- Suppose $n \notin E$
- Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$
- hence
$$\begin{aligned} n^2 &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \\ &= 2M + 1 \end{aligned}$$

hence $n^2 \notin E$

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by contrapositive we have shown

$n^2 \in E \Rightarrow n \in E$.

Since n was arbitrary we are done.