Equivalence

Given P, Q the statement $P \Leftrightarrow Q$
(read: "P if and only if Q"
"P iff Q")
is true if and only if P, Q have
same truth value.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

1. $1+1=2 \Leftrightarrow 2+2=4$ is (T)
2. $1+1=3 \Leftrightarrow 2+2=5$ is (T)
3. $1+1=2 \Leftrightarrow 2+2=5$ is (F)
4. $(\forall x \in \mathbb{R})(x \geq 0 \Leftrightarrow (\exists y \in \mathbb{R})(x = y^2))$ is (T)

↳ Why: for every $x \in \mathbb{R}$, the statements "$x \geq 0$" and "$(\exists y \in \mathbb{R})(x = y^2)$" are either both true or both false

Def'n Statements P, Q are logically equivalent iff $P \Leftrightarrow Q$ is true.

↳ e.g. $1+1=2$ and $2+2=4$ are logic. equiv.

→ more interested in logically equivalent forms for connected statements, esp. negated statements.

Negation of Quantified Statements

- Sps $P(x)$ is a var prop'n and S is a ~~set~~ nonempty set.
- Consider the negated statements

① $\neg(\forall x \in S)\, P(x)$
② $\neg(\exists x \in S)\, P(x)$

Observe: ① is true iff there is $x \in S$ s.t. $P(x)$ is false, i.e. iff
$(\exists x \in S)\, \neg P(x)$
is true.
② is true iff for every $x \in S$, $\neg P(x)$ is true, i.e. iff
$(\forall x \in S)\, \neg P(x)$ is true.

More succinctly, we have that

$$\neg(\forall x \in S)\, P(x) \iff (\exists x \in S)\, \neg P(x)$$

is true (i.e. $\neg(\forall x \in S)\, P(x)$ and $(\exists x \in S)\, \neg P(x)$ are logically equiv.)

and similarly

$$\neg(\exists x \in S)\, P(x) \iff (\forall x \in S)\, \neg P(x)$$

is true.

<u>Ex's</u> ① $\neg(\forall x \in \mathbb{R})(x \in \mathbb{N})$ "not all reals are naturals"
is equiv to
$(\exists x \in \mathbb{R})\neg(x \in \mathbb{N})$ "there is a real which is not a natural."

↳ can write $\neg(x \in \mathbb{N})$ as $x \notin \mathbb{N}$
similarly can write $\neg(x = y)$ as $x \neq y$.

② $\neg(\exists x \in \mathbb{R})(x \in \mathbb{N}) \Leftrightarrow (\forall x \in \mathbb{R})(x \notin \mathbb{N})$
is <u>true</u> (because both indiv. statements are <u>false</u>)

"There is no real which is a natural" "every real is not a natural."

③ For multiple quantifiers: just iterate process.

$\neg(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(xy = 1)$ "not every real has a multiplicative inverse."

$\Leftrightarrow (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\neg(xy = 1)$

$\Leftrightarrow (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(xy \neq 1)$ "There is a real that has no inverse"

these are all <u>true</u> since $x = 0$ has no inverse.

Negating connected statements

Theorem. For any statements P, Q the following equivalences hold:

1. $\neg\neg P \Leftrightarrow P$
2. $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
3. $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.

to prove these equivalences, use truth tables

Pf. ①

P	$\neg P$	$\neg\neg P$	$\neg\neg P \Leftrightarrow P$
T	F	T	T
F	T	F	T

always true

②

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P\wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

...

$\neg(P\wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$
T
T
T
T

✓

③ similar

(7) and (8) are called <u>De Morgan's Laws</u> for logic.

<u>Ex's</u>

(1) $\neg\neg(1+1=2)$
is equiv. to
$1+1=2$ (both true)

(2) $\neg(1+1=2 \wedge 1+1=3)$
is equiv to (both true)
$\neg(1+1=2) \vee \neg(1+1=3)$
$\hookrightarrow$ can write as $1+1 \neq 2 \vee 1+1 \neq 3$

(3) $\neg(1+1=2 \vee 1+1=3)$
is equiv to
$1+1 \neq 2 \wedge 1+1 \neq 3$ (false)

(4) $(\forall x \in \mathbb{R})\neg(x<0 \wedge (\exists y \in \mathbb{R})(y^2=x))$

$\iff (\forall x \in \mathbb{R})[\neg(x<0) \vee \neg(\exists y \in \mathbb{R})(y^2=x)]$

$\iff (\forall x \in \mathbb{R})[(x \geq 0) \vee (\forall y \in \mathbb{R})(y^2 \neq x)]$

(true)

Other useful logical equivalences

<u>Thm</u>: the following equivalency hold:

① $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$
② $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
③ $(P \Leftrightarrow Q) \Leftrightarrow (P \Rightarrow Q \wedge Q \Rightarrow P)$

we'll use these in proofs!

<u>Pf</u>: ① and ②:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T	T
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$	$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$
T	T
T	T
T	T
T	T

...

③ Try it.

<u>Thm</u>: the following equivalency hold.

① $\neg(P \Rightarrow Q) \Leftrightarrow (P \wedge \neg Q)$
② $\neg(P \Leftrightarrow Q) \Leftrightarrow [(P \wedge \neg Q) \vee (\neg P \wedge Q)]$

Pf: instead of a table, we can use our previous equivalences.

① $\neg(P \Rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q)$
$\Leftrightarrow \neg\neg P \wedge \neg Q$
$\Leftrightarrow P \wedge \neg Q$ ✓

② $\neg(P \Leftrightarrow Q) \Leftrightarrow \neg[P \Rightarrow Q \wedge Q \Rightarrow P]$
$\Leftrightarrow \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)]$
$\Leftrightarrow \neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)$
$\Leftrightarrow (\neg\neg P \wedge \neg Q) \vee (\neg\neg Q \wedge \neg P)$
$\Leftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P)$

"transitivity of $\Leftrightarrow$"

Ex's: Let E, O, P denote the sets of even, odd, and prime positive integers, resp.

① $5 \in O \Rightarrow 6 \in E$
is equiv to
$\neg(5 \in O) \vee 6 \in E$
which we can write
$5 \notin O \vee 6 \in E$ (True)

② $(\forall x \in \mathbb{N})(x \in O \Rightarrow x+1 \in E)$
equiv to
$(\forall x \in \mathbb{N})(x \notin O \vee x+1 \in E)$
equiv to
$(\forall x \in \mathbb{N})(x+1 \notin E \Rightarrow x \notin O)$ (True)

3 $(\forall x \in \mathbb{N})(x \in P \Leftrightarrow x \in O)$

is equiv. to

$(\forall x \in \mathbb{N})[(x \in P \Rightarrow x \in O) \wedge (x \in O \Rightarrow x \in P)]$

(False)

4 Consider the (true) statement

$$(\forall x \in \mathbb{R})\left[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(x = y^2)\right]$$

Let's find its logical negation in positive form:

$\neg(\forall x \in \mathbb{R})\left[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(x = y^2)\right]$

$\Leftrightarrow (\exists x \in \mathbb{R})\neg\left[(x \geq 0) \Leftrightarrow (\exists y \in \mathbb{R})(x = y^2)\right]$

$\Leftrightarrow (\exists x \in \mathbb{R})\left[((x \geq 0) \wedge \neg(\exists y \in \mathbb{R})(x = y^2)) \vee ((x < 0) \wedge (\exists y \in \mathbb{R})(x = y^2))\right]$

$\Leftrightarrow (\exists x \in \mathbb{R})\left[((x \geq 0) \wedge (\forall y \in \mathbb{R})(x \neq y^2)) \vee (x < 0) \wedge (\exists y \in \mathbb{R})(x = y^2))\right]$ ✓

Def'n A statement P is in positive form if any negation symbols in P occur next to substatements that contain no connectives or quantifiers

↳ rules above allow you to find for any P a logically equivalent P' in positive form.

<u>Thm</u> (Associative + distributive laws)
The following equivalences hold:

1. $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$
2. $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

3. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$
4. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

For proofs see 4.6.3 and 4.6.4 in textbook.

<u>Proving equality of sets using ⇔'s</u>

– there is a strong analogy between logical connectives and set operations introduced in ch 2

<u>Connective</u>	<u>Operation</u>
$P \wedge Q$	$A \cap B$
$P \vee Q$	$A \cup B$
$P \Rightarrow Q$	$A \subseteq B$
$P \Leftrightarrow Q$	$A = B$
$\neg P$	$\overline{A}$

– analogy gives us a new way of proving equality of two sets, by a string of ⇔'s.

<u>Theorem</u> Suppose A, B are sets and U is a universal set with $A, B \subseteq U$.
Then we have:

<u>looks like:</u>

① $\bar{\bar{A}} = A$ $\qquad \neg\neg P \Leftrightarrow P$

② $\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\qquad \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

③ $\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\qquad \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

<u>Pf:</u> ① Fix $x \in U$ $\longleftarrow$ not in A or $\bar{A}$!!

then
$$\begin{aligned} x \in \bar{\bar{A}} &\Leftrightarrow x \notin \bar{A} && \text{defn of complement} \\ &\Leftrightarrow \neg(x \in \bar{A}) \\ &\Leftrightarrow \neg(\neg(x \in A)) && \text{defn of complemt} \\ &\Leftrightarrow x \in A && \neg\neg P \Leftrightarrow P \end{aligned}$$

thus chain of equivalences shows
$$x \in \bar{\bar{A}} \Leftrightarrow x \in A$$
i.e.
$$x \in \bar{\bar{A}} \Rightarrow x \in A \qquad (\bar{\bar{A}} \subseteq A)$$
<u>and</u>
$$x \in A \Rightarrow x \in \bar{\bar{A}} \qquad (A \subseteq \bar{\bar{A}})$$

hence we've <u>proved</u> $\bar{\bar{A}} = A$. ✓

② Fix $x \in U$

then
$$\begin{aligned} x \in \overline{A \cap B} &\Leftrightarrow x \notin A \cap B && \text{defn of comp} \\ &\Leftrightarrow \neg(x \in A \cap B) \\ &\Leftrightarrow \neg(x \in A \wedge x \in B) && \text{defn of } \cap \\ &\Leftrightarrow \neg(x \in A) \vee \neg(x \in B) && \text{DeMorgan} \\ &\Leftrightarrow x \in \bar{A} \vee x \in \bar{B} \\ &\Leftrightarrow x \in \bar{A} \cup \bar{B} \end{aligned}$$

(3) similar.

Exercise: use the distributive law $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ to prove:

Theorem For any sub A, B, C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof writing

Two approaches: - when trying to prove a statement P, can either prove P directly, or assume $\neg P$ and derive a contradiction (i.e. prove $\neg\neg P$)

- more generally: can prove any statement logically equiv. to P, or disprove any statement logically equiv. to $\neg P$.

Existence Proofs

General form $(\exists x \in S)\, P(x)$

Direct Proof strategy: define an el't $y \in S$ and prove $P(y)$ holds.

Ex ① Prop'n There is an even number that can be written as the sum of two primes in two distinct ways

$24 = 19 + 5$
$= 17 + 7$

Pf: - Consider $n = 10$.
- Then n is even and we have $n = 5+5$ and $n = 7+3$
- since 3, 5, 7 are primes the prop'n is proved!

note: $n = 24 = 19+5 = 17+7$ works too..

Indirect Proof Strategy:

- Assume $\neg(\exists x \in S)P(x)$ and derive a contradiction
- equivalently, assume $(\forall x \in S)\neg P(x)$ and get a contradiction

Ex ② Fix $n \in \mathbb{N}$ and suppose $a_1, \dots, a_n \in \mathbb{R}$. Then at least one of $a_1, \dots, a_n$ is at least as large as their average.

That is:

$$(\exists k \in [n])\left(a_k \geq \underbrace{\tfrac{1}{n}(a_1 + \cdots + a_n)}_{= \frac{1}{n}\sum_{i=1}^{n} a_i}\right)$$

Secretly a universal claim. we focus on existential part.

Pf. - Suppose not, toward a contradiction

- that is: suppose that

$$(\forall k \in [n])\left(a_k < \tfrac{1}{n}(a_1 + \cdots + a_n)\right)$$

- For simplicity let $S = a_1 + \cdots + a_n$
- our assumption is, for every $k \in [n]$ we have

$$a_k < S/n$$

But then we have

$$\begin{aligned} S &= a_1 + a_2 + \cdots + a_n \\ &< S/n + S/n + \cdots + S/n \quad \text{by our assumption} \end{aligned}$$

$= n(s/n)$
$= s$

- This shows $s<s$, a contradiction
- Thus our assumption was false, and hence the prop'n must be true. ✓

Universal Proofs

General form: $(\forall x \in S)\, P(x)$

Direct Strategy:
- Let $x \in S$ be arbitrary but fixed
- Prove $P(x)$ holds

Ex ① Prop'n $(\forall x,y \in \mathbb{R})\left(xy \leq \left(\frac{x+y}{2}\right)^2\right)$

Pf: - Fix $x, y \in \mathbb{R}$.
- Since squares are always non-negative we have
$0 \leq (x-y)^2$
- hence
$0 \leq x^2 - 2xy + y^2$
- hence
$2xy \leq x^2 + y^2$
i.e. $xy \leq (x^2+y^2)/2$

- Since x, y were arbitrary the prop'n is proved.

Note: prop'n is one version of "AMGM" Inequality

↳ arithmetic mean (AM) of x,y is $x+y/2$

↳ geometric mean (GM) of $x,y \geq 0$ is $\sqrt{xy}$

prop'n gives for $x,y \geq 0$

$$\sqrt{xy} \leq x+y/2$$

i.e. $GM \leq AM$.

Indirect Strat:

- Assume $\neg(\forall x \in S)\, P(x)$ (i.e. $(\exists x \in S)\, \neg P(x)$) and get a contradiction

Ex ② Prop'n $\sqrt{2}$ is irrational, that is,

$$(\forall a, b \in \mathbb{Z})\left(\frac{a}{b} \neq \sqrt{2}\right)$$

Pf: - Suppose not, that is, suppose $\exists a, b \in \mathbb{Z}$ s.t.

$$a/b = \sqrt{2}$$

- We may assume a/b is in reduced form, i.e. a and b have no common factors since if they do we can cancel and get a fraction in reduced form.