# Ch. 4: Intro to Mathematical Logic

**Goals:** - Learn how to write more formal statements and proofs

- expand repertoire of proof techniques

→ more symbols, fewer words

Recall: Def'n (intuitive): A mathematical statement (or prop'n) is a grammatically correct declarative sentence that is either true or false.

→ may consist of words and/or symbols

- to rigorously define "statement" requires more formal logic - these statements are entirely symbolic.

- "grammatically correct" has a precise meaning in that context.

**Ex's** ① Every integer is a real number (T)

② Every real number is an integer (F)

3. There exists $x \in \mathbb{R}$ s.t. $x \notin \mathbb{Z}$ (T)
4. $1+2=3$ (T)
5. Every integer greater than 5 can be written as the sum of three primes (<u>unknown</u>: but either T or F)

<u>Nonex's</u>
1. $\phi\, 3\, \pi$ (not grammatically correct/meaningless)
2. Shakespeare (not a declarative sentence/no truth value)
3. $x^2+1=2$

↳ meaningful sequence of symbols, but no truth value unless $x$ is specified

- called a <u>variable proposition</u>: a sentence that becomes a statement once its variables are specified (or quantified over... more later)

- We'll use $P, Q, R, \ldots$ for statements and $P(x), Q(x,y), \ldots$ for var. prop'ns.

eg. might say:

- Let $P$ be the statement "$5^2+1=2$" (F)
- Let $Q(x)$ be the var. prop'n "$x^2+1=2$"

Then $Q(5)$ denotes the statement
"$5^2+1=2$" (F)
And $Q(1)$ denotes
"$1^2+1=2$" (T)

Ex's of var. prop'ns
1. $x^2+1 \leq 0$
2. $x \in \mathbb{R}$ and $x<3$
3. $z = x^2+y$

↳ multiple variables:
indicate when abbreviating

e.g. might use $Q(x,y,z)$ to denote 3.

Then $Q(1,2,3)$ is F
but $Q(2,1,1)$ is T.

Quantifying variables

↳ other way to turn a var prop'n into a statement is to quantify over its variables

e.g. "$x^2+1=2$" is a var. prop'n
but
"There exists $x \in \mathbb{R}$ s.t. $x^2+1=2$"
is a statement (T)

as is
"For every $x \in \mathbb{R}$ we have $x^2+1=2$"
(F)

The clauses "There exists $x \in S, \ldots$" and "For every $x \in S, \ldots$" are examples of quantification of the variable $x$.

We introduce the symbols:

$\forall$ read: "for all"

$\exists$ read: "there exists"

$\forall$ is called the universal quantifier

$\exists$ " " existential quantifier

Given a var prop'n $P(x)$ and a set $S$

"For every $x \in S$ we have $P(x)$"

"There exists $x \in S$ s.t. $P(x)$"

are statements

Write them symbolically as:

$(\forall x \in S)\, P(x)$ or $\forall x \in S.\, P(x)$

$(\exists x \in S)\, P(x)$ or $\exists x \in S.\, P(x)$

<u>Ex's</u> ① $(\exists x \in \mathbb{N})\,(x<5)$

"There is a natural number less than 5" (T)

2. $(\forall x \in \mathbb{N})(x < 5)$
"Every natural number is less than 5" (F)

3. $(\forall x \in \mathbb{N})(x > 0)$ (T)
4. $(\forall x \in \mathbb{Z})(x > 0)$ (F)

→ truth of a statement depends on set we quantify over.

## Multiple quantifiers

5. $(\forall x, y \in \mathbb{N})(x + y \geq 2)$
read: "For every $x, y$ in $\mathbb{N}$ we have $x+y \geq 2$" (T)

- can also nest $\forall$'s and $\exists$'s but beware: order is important!

6. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{R})(x = y^2)$
"For every $x \in \mathbb{N}$ there is $y \in \mathbb{R}$ s.t. $x = y^2$"
i.e. every natural # has a real square root (T)

7. $(\forall x \in \mathbb{R})(\exists y \in \mathbb{N})(x = y^2)$
i.e. every real has a square root in $\mathbb{N}$ (F)

- What happens if we reverse the order of quantifiers in (e).
  ↳ completely changes meaning:

$(\exists y \in \mathbb{R})\ (\forall x \in \mathbb{N})\ (x = y^2)$

"There is a real number $y$ s.t. every natural number equals $y^2$"

↳ perfectly well-written statement but absurd and definitely <u>false</u>.

<u>moral</u>: order of quantifiers a big <u>deal</u>!

(8) "Inside" quantifiers: following is also a well-written sentence

$(\forall x \in \mathbb{R})\ (\text{if } x \geq 0, \text{ then } (\exists y \in \mathbb{R})(y^2 = x))$

(T)

<u>Note</u>: quantifying set variables

- we've insisted all quantified variables range over a specified set, e.g.

$(\forall x \in \mathbb{R})\ (x^2 \geq 0)$ is meaningful

$(\forall x)\ (x^2 \geq 0)$ is <u>not</u>

- what if we want our variables to stand for <u>sets</u>?

- e.g. if we wanted to write
  "For every set $S$, $\emptyset \subseteq S$"
symbolically, might try:

$$(\forall S \in (\cdots))(\emptyset \subseteq S)$$

set of all sets??

- but the collection of all sets is not a set (Russell's paradox)

Convention: when quantifying set variables, we'll write sentences verbally

i.e. "For every set $S$ ..."
"There exists a set $S$ ..."

## Connectives + Truth Tables

- Connectives are symbols used to combine multiple statements into one.
- all our connectives will be binary (for combining two statements into one) except negation, which is unary
- Truth Tables tell us how truth of connected statements depends on truth of the original statements

## Conjunction ("and")

- the conjunction of statements $P, Q$ is written $P \wedge Q$ ("P and Q")
- $P \wedge Q$ is true iff both $P, Q$ are true

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Ex's ① - Let $P$ be
$$(\forall x \in \mathbb{R})(x+1 > x)$$
- Let $Q$ be
2 is a prime number
- let $R$ be
$$2^2 = 5$$

Then $P, Q$ are T
$R$ is F

Hence $P \wedge Q$ is T
but $P \wedge R$ and $Q \wedge R$ are both F

Written out, $P \wedge Q$ is
$$(\forall x \in \mathbb{R})(x+1 > x) \wedge (2 \text{ is prime})$$

Sometimes inserting some extra parentheses clarifies expression

## Disjunction ("or")

- the disjunction of P, Q is written $P \vee Q$ ("P or Q")
- is true iff at least one of P, Q is true

| P | Q | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

e.g.

$$\underbrace{(\forall x \in \mathbb{R})(x^2 \geq 0)}_{T} \vee \underbrace{(4 \text{ is prime})}_{F}$$

is true, but

$$\underbrace{(4 \text{ is prime})}_{F} \vee \underbrace{(6 \text{ is prime})}_{F}$$

is false.

## Negation

- only unary connective
- negation of P written $\neg P$
- true iff P is false

| P | $\neg P$ |
|---|---|
| T | F |
| F | T |

Ex's ① $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$
is false, hence
② $\neg(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$
is true, while
③ $\neg\neg(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y^2 = x)$
is false again

④ For any statement P,
the statement $P \vee \neg P$ is true
e.g.
(6 is prime) $\vee \neg$ (6 is prime)
is true.

More examples

⊕ Can also use connectives in variable propositions
– as before var prop'ns become T or F only once variables are specified

e.g. let $P(x,y)$ be

$(x > 0) \wedge (y \text{ is prime})$

then $P(3,5)$ is true
while $P(3,6)$ is False

and $(\forall x, y \in \mathbb{N})\, P(x,y)$ is false as well

(written out:
$(\forall x, y \in \mathbb{N})((x > 0) \wedge (y \text{ is prime}))$

② $(\forall x \in \mathbb{R})\,(x < 0 \vee (\exists y \in \mathbb{R})(x = y^2))$

is ... <u>true</u>.

③ - can use connectives in def'ns, set-builder notation, etc...
- e.g. if A, B are subsets of a universal set U then

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\}$$
$$A \cup B = \{x \in U \mid x \in A \vee x \in B\}$$

- explore connection between set operations and connectives more later.

## <u>Implication</u>

- Given statements P, Q the statement $P \Rightarrow Q$ is read "P implies Q" or "if P, then Q".
- $P \Rightarrow Q$ is true iff
  <u>if</u> P is true, Q is <u>also</u> true.

| P | Q | $P \Rightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

- most confusing connective
- notice: $P \Rightarrow Q$ is always true when $P$ is false
- $P \Rightarrow Q$ is only false when $P$ is true but $Q$ is false.

- statements of form $P \Rightarrow Q$ are called conditional statements

Ex's ① $\overset{T}{1+1=2} \Rightarrow \overset{T}{1+1+1=3}$
is true
② $1+1=2 \Rightarrow 1+1+1=4$
is false
③ $1+1=2 \Rightarrow \pi \notin \mathbb{Q}$
is also true even though P, Q are unrelated.
④ $(\exists x \in \mathbb{R})(x^2=-1) \Rightarrow 1+1=3$
is true! automatically because P in this case is F

$(\exists x \in \mathbb{R})(x^2=-1) \Rightarrow 1+1=2$
also true.

⑤ can use $\Rightarrow$ in var prop'ns
e.g.
$x \geq 2 \Rightarrow x^2 \geq 4$
is a well-formed var prop'n
and
$(\forall x \in \mathbb{R})(x \geq 2 \Rightarrow x^2 \geq 4)$
is a true statement.

OTOH

6 $(\forall x \in \mathbb{R})\ (x^2 \geq 4 \Rightarrow x \geq 2)$

is <u>false</u>

because there is a real number $x$ (e.g. $x = -3$) s.t. $x^2 \geq 4$ ~~[illegible]~~ is true but $x \geq 2$ is false.