Powersets

- Consider the set $A = \{1,2,3\}$
- Can we list all subsets of A?

Sure:

$\emptyset$ $\{1\}$, $\{2\}$, $\{3\}$ $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ $\{1,2,3\}$

- the set of all these subsets is called the powerset of A.

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Def'n Given a set X, the powerset of X, denoted $P(X)$ is the set of all subsets of X

i.e.

$Y \in P(X)$ iff $Y \subseteq X$.

Ex's ① $\{1,2,3\} \subseteq \mathbb{N}$

hence $\{1,2,3\} \in P(\mathbb{N})$

also $E \in P(\mathbb{N})$ and $O \in P(\mathbb{N})$

but $\{-1,0,1\} \notin P(\mathbb{N})$

because $\{-1,0,1\} \not\subseteq \mathbb{N}$

however $\{-1,0,1\} \in P(\mathbb{Z})$

② Prop'n if A, B are sets and $A \subseteq B$, then $P(A) \subseteq P(B)$

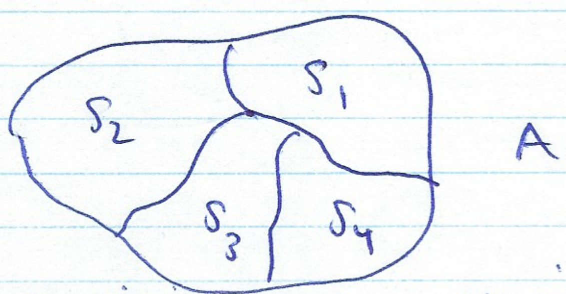
Pf: - Fix an arbitrary $X \in P(A)$
- then by def'n of powerset we have $X \subseteq A$
- since $X \subseteq A$ and $A \subseteq B$, by transitivity of $\subseteq$ we have $X \subseteq B$
- Hence $X \in P(B)$
- Since $X \in P(A)$ was arbitrary we have that for every such X that $X \in P(B)$
- Hence $P(A) \subseteq P(B)$ ✓

(3) What is $P(\phi)$?
Only subset of ϕ is ϕ
Hence $P(\phi) = \{\phi\}$.

Partitions

A partition of a set A is a collection of subsets of A that split up A into disjoint pieces.

Pic:



a partition into 4 pieces

Formal def'n: Let A be a set.
A partition of A is a collection (i.e. set) of nonempty sets S_i, $i \in I$ such that

(i) for every $i \in I$ we have $S_i \subseteq A$

(ii) if $i \neq j$, then $S_i \cap S_j = \emptyset$

(iii) $\bigcup_{i \in I} S_i = A$.

vocab: (ii) says that the sets S_i are "pairwise disjoint."

Ex's ① Let $A = \{1,2,3,4,5,6\} = [6]$
Let $S_1 = \{1\}$
Let $S_2 = \{2,3,4\}$
$S_3 = \{5,6\}$

Then $\{S_1, S_2, S_3\} = \{\{1\}, \{2,3,4\}, \{5,6\}\}$ is a partition of A

Why: (i) S_1, S_2, S_3 all (nonempty) subsets of A ✓

(ii) $S_1 \cap S_2 = S_1 \cap S_3 = S_2 \cap S_3 = \emptyset$ ✓

(iii) $S_1 \cup S_2 \cup S_3 = A$ ✓

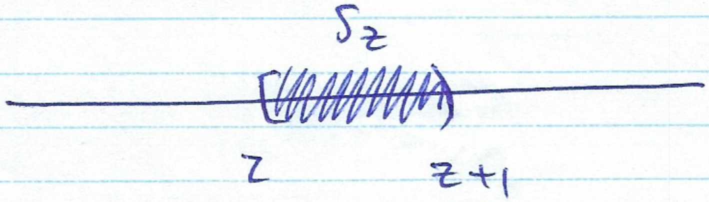
However: – $\{S_1, S_2\}$ is not a partition of A (condition (iii) fails)

– Let $S_4 = \{1,6\}$
Then $\{S_1, S_2, S_3, S_4\}$ is not a partition either since $S_1 \cap S_4 = \{1\} \neq \emptyset$.

(2) Sometimes we drop the indices, e.g.
$\{E, O\}$ is a partition of $\mathbb{N}$.

(3) - Consider the set $\mathbb{R}$
- For every $z \in \mathbb{Z}$ define

$$S_z = \{x \in \mathbb{R} \mid z \leq x < z+1\}$$
$$= [z, z+1)$$



Can you see why $\{S_z : z \in \mathbb{R}\}$ is a partition of $\mathbb{R}$?

(4) OTOH if we define

$$T_z = \{x \in \mathbb{R} \mid z \leq x \leq z+1\}$$
$$= [z, z+1]$$

then $\{T_z : z \in \mathbb{Z}\}$ is __not__ a partition of $\mathbb{R}$, since e.g.
$$T_1 \cap T_2 = [1,2] \cap [2,3]$$
$$= \{2\} \neq \emptyset.$$

Cartesian Products

- $\mathbb{R}\times\mathbb{R}$ = set of ordered pairs of real numbers
  $= \{(a,b) \mid a\in\mathbb{R} \text{ and } b\in\mathbb{R}\}$

  technically an illegal use of set-builder notation

Formally:

Def'n: Suppose A, B are sets. The Cartesian Product of A, B, denoted $A\times B$, is the set of all ordered pairs (a,b) with $a\in A$ and $b\in B$.

$$A\times B = \{(a,b) \mid a\in A \text{ and } b\in B\}$$

Note: - order is important! First coord. from A, second from B. Could have $A\neq B$.

- sometimes write A^2 for $A\times A$.

Ex's ① if $A=\{1,2,3\}$
$B=\{*,♡\}$
then $A\times B = \{(1,*),(2,*),(3,*),(1,♡),(2,♡),(3,♡)\}$

2 - $(1, \pi)$ and $(2, \sqrt{2})$ are el'ts of $\mathbb{N} \times \mathbb{R}$
- $(\pi, \sqrt{2}) \notin \mathbb{N} \times \mathbb{R}$ since $\pi \notin \mathbb{N}$

3 Triples: Given sets A, B, C can define
$$A \times B \times C = \{(a,b,c) \mid a \in A, b \in B, c \in C\}$$

- e.g. $(1, \pi, 2+i) \in \mathbb{N} \times \mathbb{R} \times \mathbb{C}$.

- $A \times B \times C$ is not the same as $(A \times B) \times C$: el'ts of first set look like (a,b,c); el'ts of the second look like $((a,b),c)$

- but these two sets are "essentially" the same.

4 Can use pairs, triples, etc. in set-builder notation
e.g. if we write
$X = \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n \in E \text{ and } m \in O\}$
then X is the set of pairs of natural numbers where first coord. is even and second coord. is odd.

So: $(2, 7) \in X$
but $(4, 8) \notin X$.

Proofs with sets

Proving $A \subseteq B$

Strategy:
1. Fix an arbitrary $a \in A$
2. Prove $a \in B$
3. Conclude ~~[illegible]~~, since a was arbitrary, that for every $x \in A$ we have $x \in B$, i.e. $A \subseteq B$.

Ex 1 **Prop'n** Suppose A, B, X are sets. If $X \subseteq A$ and $X \subseteq B$, then $X \subseteq A \cap B$.

Pf:
- Fix an arbitrary $x \in X$
- Since ~~[illegible]~~ $X \subseteq A$, we have $x \in A$
- Since $X \subseteq B$, we have $x \in B$ also.
- Hence $x \in A \cap B$.
- Since $x \in X$ was arbitrary, we have proved that for every $y \in X$ we have $y \in A \cap B$, i.e. $X \subseteq A \cap B$ ✓

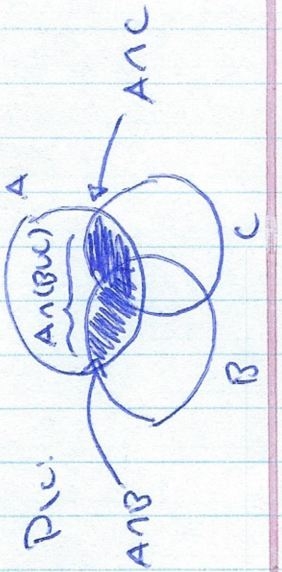
2 **Prop'n** Suppose A, B are sets. Then $P(A) \cap P(B) \subseteq P(A \cap B)$

Pf:
- Fix an arbitrary $X \in P(A) \cap P(B)$
- Then $X \in P(A)$ and $X \in P(B)$ (by def'n of $\cap$)
- Hence $X \subseteq A$ and $X \subseteq B$ (by def'n of powerset)
- Hence by our previous prop'n, $X \subseteq A \cap B$, i.e. $X \in P(A \cap B)$
- Since $X \in P(A) \cap P(B)$ was arbitrary we have $P(A) \cap P(B) \subseteq P(A \cap B)$ ✓

3c

<u>Proving $A = B$</u>

Strategy:
1. Show $A \subseteq B$
2. Show $B \subseteq A$
3. Conclude $A = B$.



<u>Ex</u> ① <u>Prop'n</u>: Let A, B, C be sets. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

<u>Pf</u>: ($\subseteq$) – fix $x \in A \cap (B \cup C)$

– then: (i) $x \in A$ <u>and</u>
(ii) $x \in B \cup C$

– <u>Case 1</u>: $x \in B$. In this case, since $x \in A$, we have $x \in A \cap B$.

– <u>Case 2</u>: $x \notin B$. In this case we must have $x \in C$, by (ii), since $x \in B$ <u>or</u> $x \in C$.

↳ Hence $x \in A \cap C$ in this case.

– Thus either $x \in A \cap B$ <u>or</u> $x \in A \cap C$
i.e. $x \in (A \cap B) \cup (A \cap C)$

– Since x was arbitrary we have
$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

($\supseteq$) – now fix $x \in (A \cap B) \cup (A \cap C)$

– <u>Case 1</u>: $x \in A \cap B$.
– in this case, $x \in A$ <u>and</u> $x \in B$
– hence $x \in A$ <u>and</u> ($x \in B$ or $x \in C$)
– i.e. $x \in A \cap (B \cup C)$

– <u>Case 2</u>: $x \notin A \cap B$
– then since $x \in A \cap B$ or $x \in A \cap C$ we must have $x \in A \cap C$.
– i.e. $x \in A$ <u>and</u> $x \in C$

- hence $x \in A$ and ($x \in B$ or $x \in C$)
- hence $x \in A \cap (B \cup C)$

Thus in all cases $x \in A \cap (B \cup C)$
Since x was arbitrary we have
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

We have shown already
$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Hence these sets are equal. ✓

Counterexamples

To show a statement is false
sufficient to provide a single
counterexample.

__Ex__: Disprove the following:
For any sets A, B, C, if
$A \subseteq B \cup C$ then either $A \subseteq B$ or
$A \subseteq C$.

__Sol'n__: consider the sets:

$$A = \{2,3\}$$
$$B = \{1,2\}$$
$$C = \{3,4\}$$

then:
$A = \{2,3\} \subseteq \{1,2,3,4\} = B \cup C.$

but $A \not\subseteq B$ and $A \not\subseteq C$
hence the statement is false.

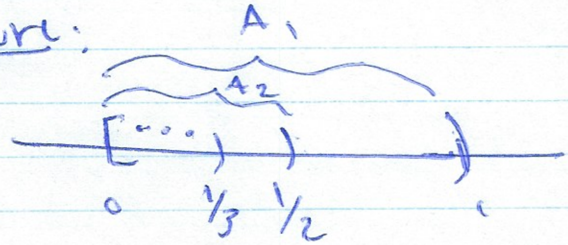
Using Contradiction

idea: to prove that a statement S is true, you can
① Assume S is false
② Show this assumption contradicts your hypothesis (or a previously established statement)
③ Conclude S is true.

Ex: Prop'n For every $n \in \mathbb{N}$, define
$A_n = \{x \in \mathbb{R} \mid 0 \leq x < 1/n\}$

Then:
$$\bigcap_{n \in \mathbb{N}} A_n = \{0\}$$

Picture:



Pf: We show $\supseteq$ first, since this is easier.

($\supseteq$) – To show $\{0\} \subseteq \bigcap_{n\in\mathbb{N}} A_n$, sufficient to show $0 \in \bigcap_{n\in\mathbb{N}} A_n$

– by def'n, $0 \in \bigcap_{n\in\mathbb{N}} A_n$ iff $0 \in A_n$ for every $n \in \mathbb{N}$.

– Fix $n \in \mathbb{N}$, Then of course $0 \in A_n = \{x \in \mathbb{R} \mid 0 \leq x < 1/n\}$ since indeed $0 \leq 0 < 1/n$.

– Since $n \in \mathbb{N}$ was arbitrary, we have $0 \in A_n$ for every $n \in \mathbb{N}$.

– i.e. $0 \in \bigcap_{n\in\mathbb{N}} A_n$ ✓

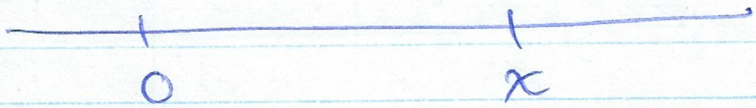
($\subseteq$) – We now show $\bigcap_{n\in\mathbb{N}} A_n \subseteq \{0\}$

– Toward a contradiction, suppose not. Then there must be some $x \in \bigcap_{n\in\mathbb{N}} A_n$ s.t. $x \notin \{0\}$

– i.e. $x \neq 0$

can it have $x<0$!

– Since for every $n \in \mathbb{N}$ and every $y \in A_n$ we have $y \geq 0$, we must actually have $x > 0$

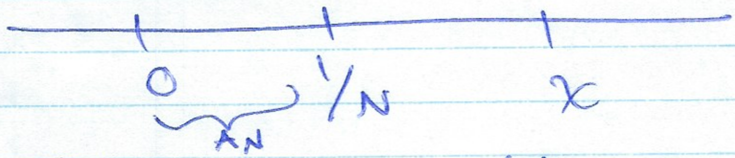


(the trick): – Consider $1/x$

– if x is very small, $1/x$ may be very large.

– however there is <u>some</u> N s.t. $N > 1/x$

– hence $1/N < x$



– But then $x \notin A_N$, by def'n of A_N

– This is a contradiction, since we assumed $x \in \bigcap_{n \in N} A_n$.

– hence it must be that

$$\bigcap_{n \in N} A_n \subseteq \{0\}$$

– Thus

$$\bigcap_{n \in N} A_n = \{0\}.$$