(i) picking the k people
(ii) from them, choosing a chair

so: $|S| = \binom{n}{k} \cdot \binom{k}{1}$
$= k\binom{n}{k}$

- or could form a committee by
(i) first picking the chair
(ii) from remaining n-1 people, choose k-1 other members of comm.

So $|S| = n\binom{n-1}{k-1}$

so: $k\binom{n}{k} = \binom{n-1}{k-1}$ ✓

We can verify identity algebraically:

$$\binom{n}{k}k = \frac{n!}{k!(n-k)!}k = \frac{n!}{(k-1)!(n-k)!}$$

OTOH

$$n\binom{n-1}{k-1} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}$$

$$= \frac{n!}{(k-1)!(n-k)!}$$ ✓

Ex: Prop'n Fix $n \in \mathbb{N}$. Then:

$$n 2^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k$$

Pf: Let S be the set of nonempty committees w/ a chairperson that can be chosen from a group of n people.

Done ✓

But let's check:

Can form such a committee by
(i) choosing the chair
(ii) from remaining $n-1$ ppl choosing other committee members (this is just choosing a subset from a set of size $n-1$)

S: $|S| = n \cdot 2^{n-1}$

OTOH: we can partition S:

$$S = A_1 \cup A_2 \cup \cdots \cup A_n$$

where A_k is the set of such committees consisting of exactly k people.

above we computed

$$|A_k| = \binom{n}{k} k$$

Hence
$$\begin{aligned} |S| &= |A_1| + |A_2| + \cdots + |A_n| \\ &= \binom{n}{1} 1 + \binom{n}{2} \cdot 2 + \cdots + \binom{n}{n} n \\ &= \sum_{k=1}^{n} \binom{n}{k} k \end{aligned}$$

and as a bonus identity, we can use our prev. example to write this as:

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} \quad \checkmark$$

Inclusion / Exclusion

↳ ROS says if we can partition a set A as:

$$A = A_1 \cup A_2 \cup \dots \cup A_k$$

then $|A| = |A_1| + |A_2| + \dots + |A_k|$
$= \sum_{i}^{k} |A_i|$

But what if ~~...~~

$$A = A_1 \cup \dots \cup A_k$$

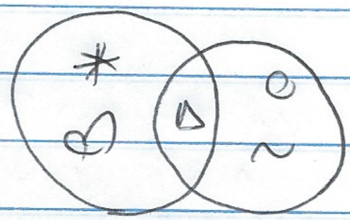
but the A_i are not pairwise disjoint? Can we still count $|A|$ in terms of $|A_i|$'s?

Yes! But harder, need the principle of inclusion / exclusion

ex: let $A_1 = \{*, \heartsuit, \Delta\}$
$A_2 = \{\Delta, \circ, \sim\}$

let $A = A_1 \cup A_2$

what is $|A|$?



A_1 A_2

is $|A| = |A_1| + |A_2|$? not quite because $\Delta \in A_1$ and $\Delta \in A_2$

but we can think of counting $|A|$ as $|A_1| + |A_2|$ then correcting overcounting

In this case we count Δ twice
so: $|A| = |A_1| + |A_2| - 1$
$= 3 + 3 - 1 = 5$ ✓

and indeed $A = A_1 \cup A_2 = \{*, \heartsuit, \Delta, \circ, \sim\}$

in general: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

e.g. if $A_1 = \{1,3,5,7\}$ $A_2 = \{2,3,5,8\}$
then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$
$$= 4 + 4 - 2 = 6$$

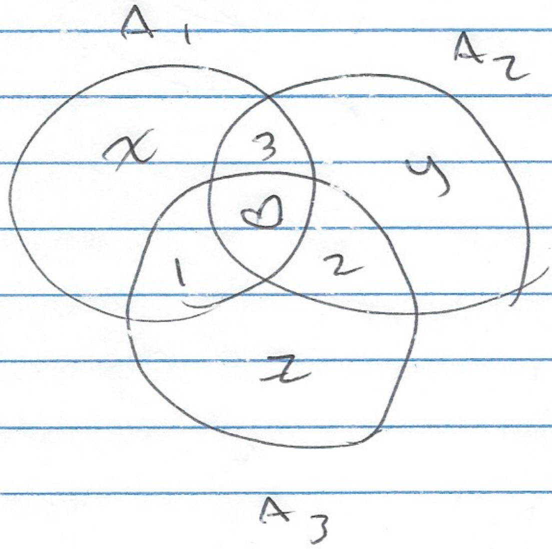
and indeed

$$A_1 \cup A_2 = \{1,2,3,5,7,8\}$$

What about 3 sets?

ex: Sps $A_1 = \{x, 1, 3, \heartsuit\}$
$A_2 = \{y, 2, 3, \heartsuit\}$
$A_3 = \{z, 1, 2, \heartsuit\}$

What is $|A_1 \cup A_2 \cup A_3|$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$
$$- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$$

elt's here counted twice

$$+ |A_1 \cap A_2 \cap A_3|$$

elt's here counted twice, subtracted three times, so need to add on

Reasoning like this more generally gives:

Thm (principle of inclusion/exclusion — PIE)
If

$$A = A_1 \cup A_2 \cup \cdots \cup A_k$$

Then $|A| = |A_1 \cup \cdots \cup A_k|$

$$= \sum_{i=1}^{k} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots (-1)^{k-1} |A_1 \cap \cdots \cap A_k|$$

= (sum over all $\binom{k}{1}$ single ~~sets~~ intersections)
− (sum over all $\binom{k}{2}$ pairs of ~~sets~~)
+ (sum over all $\binom{k}{3}$ triples of sets)
−

$+(-1)^{k-1}$ (sum over all $\binom{k}{k}$ k-tuples of sets)

$$= \sum_{\emptyset \ne J \subseteq \{1, \ldots, k\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} A_j \right|$$

Ex 1: Sps 50 ppl admitted to hospital
- 30 w/ bronchitis
- 25 w/ pneumonia
- 10 w/ both

(i) How many have at least one of branch and pneum?

(ii) How many have neither?

Sol'n Let A_p = pneumonia patients
A_b = bronchitis patients

then, (i) by PIE

$$|A_p \cup A_b| = |A_p| + |A_b| - |A_p \cap A_b|$$
$$= 30 + 25 - 10$$
$$= 45$$

(ii) so # that have neither is $50 - 45 = 5$.

Ex 2 How many integers n between 1 and 1000 are not divisible by any of 5, 7, or 11?

Let A_K = set of integers between 1 and 1000 divisible by K.

So $A_5 = \{5, 10, 15, \ldots, 1000\}$

So $|A_5| = |\{5, 10, 15, \ldots, 1000\}|$

$= \lfloor \frac{1000}{5} \rfloor = 200$

Sim'ly $A_7 = \{7, 14, \ldots, 994\}$

$|A_7| = \lfloor \frac{1000}{7} \rfloor = 142$

and $|A_{11}| = \lfloor \frac{1000}{11} \rfloor = 90$

<u>Now</u>: $A_5 \cap A_7$ = set of $n \leq 1000$ divis by 5, 7

= " " by $35 = \text{lcm}(5,7)$

$= A_{35}$

So $|A_5 \cap A_7| = |A_{35}| = \lfloor \frac{1000}{35} \rfloor$

$= 28$

Sim'ly $|A_5 \cap A_{11}| = |A_{55}| = \lfloor \frac{1000}{55} \rfloor$

$= 18$

$|A_7 \cap A_{11}| = |A_{77}| = \lfloor \frac{1000}{77} \rfloor$

$= 12$

Finally $|A_5 \cap A_7 \cap A_{11}| = |A_{385}| = 2$

Hence $|A_5 \cup A_7 \cup A_{11}|$

$$= |A_5| + |A_7| + |A_{11}| - |A_5 \cap A_7| - |A_5 \cap A_{11}| - |A_7 \cap A_{11}| + |A_5 \cap A_7 \cap A_{11}|$$

$$= 200 + 142 + 90 - 28 - 18 - 12 + 2$$
$$= 376$$

We are interested in complement of this set:

\# of n w/ $1 \le n \le 1000$ divis by ~~none~~ of $5, 7, 11$ is

$$1000 - 376 = 624 \checkmark$$

Ex 3: How many integers n between 0 and $99{,}999$ inclusive have among its digits at least one of a $2, 5,$ and 8.

Sol'n: Can think of such an n as a 10-ary sequence of length 5: $\underline{0}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{5}$

So e.g. 00000 denotes 0
and 00376 denotes 376.

For $k \in \{0, 1, \ldots, 9\}$, let B_k denote the set of integers n, $0 \le n \le 99{,}999$, that have no k in their digits.

e.g. $00183 \in B_7$.

Observe: $|B_k| = 9^5$ for any $k \in \{0, \ldots, 9\}$
$=$ # of ways of forming a five digit sequence using only $\{0, 1, \ldots, k-1, k+1, \ldots, 9\}$

So in particular:
$$|B_2| = |B_5| = |B_8| = 9^5$$

Similarly for any $k, l \in \{0, \ldots, 9\}$, $k \neq l$
$$|B_k \cap B_l| = 8^5$$
$=$ # of ways of forming a five digit sequence w/o k, l.

So: $|B_2 \cap B_5| = |B_2 \cap B_8| = |B_5 \cap B_8|$
$= 8^5$

Reasoning similarly we have:

$|B_2 \cap B_5 \cap B_8| = 7^5$

Hence: $|B_2 \cup B_5 \cup B_8|$

$= 9^5 + 9^5 + 9^5$
$- 8^5 - 8^5 - 8^5$
$+ 7^5$
$= 3 \cdot 9^5 - 3 \cdot 8^5 + 7^5 = 95{,}650$

set of n's w/ at least one of 2, 5, 8 missing from its digits

Hence: # of n, $0 \le n \le 99{,}999$, w/ all of 2, 5, 8 among its digits is

$100{,}000 - 95650 = 4{,}350$

Ex: Sps an instructor has n students and wants to return HWs for peer grading s.t. no student receives their own paper. How many possible ways of doing this?

Sol'n: If we number the students $1, 2, \ldots, n$ can think of a HW return as a permutation
$$m_1, m_2, \ldots, m_n$$
where $m_i \neq i$ for all i $1 \le i \le n$

(e.g. if there are 3 students then 3 1 2 means first student gets #3's HW, second gets #1's HW, third gets #2's HW)

Let A_i be the set of perms in which $A_i = i$.

Then $|A_i| = (n-1)!$

More generally, if we have indices $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$ then

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$$

set of perms in which $i_1, i_2, \ldots, i_k$ <u>all</u> fixed.

Hence
$$\Big|\bigcup_{i \le n} A_i\Big| = \sum_{\emptyset \ne I \subset \{1, \ldots, n\}} (-1)^{|I|-1} \Big|\bigcap_{i \in I} A_i\Big|$$

$$= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots (-1)^{n-1}\binom{n}{n}0!$$

$$= \sum_{k=1}^{n} (-1)^k \binom{n}{k}(n-k)!$$

Hence # of perms where no student receives paper is

$$n - \sum_{k=1} (-1)^k \binom{n}{k}(n-k)!$$

807

(i)

Ex: Sps we draw a 7 card Hand from a standard 52 card deck.

(i) How many distinct hands are possible?

$$\binom{52}{7}$$

(ii) How many hands include no card of rank greater than 8?

↳ we are counting hands consisting only of 2's, 3's, 4's, ..., 8's

↳ there are $7 \cdot 4 = 28$ such cards

⇒ # of such hands is

$$\binom{28}{7}$$

(iii) How many hands have exactly two K's?

Form such a hand by:

- choosing 2 K's $\binom{4}{2}$
- choosing 5 remaining cards from 48 non-K's $\binom{48}{5}$

so # of hands is

$$\binom{4}{2}\binom{48}{5}$$

(ii)

(iv) How many hands contain exactly one pair (i.e. two cards of same suit, and five of other, distinct suits)

- choose rank of pair $\binom{13}{1}$
- choose pair from that rank $\binom{4}{2}$
- choose five ranks, from remaining 12 $\binom{12}{5}$
- from each of these 5 ranks, choose a card $\binom{4}{1}^5$

# of such hands is

$$\binom{13}{1}\binom{4}{2}\binom{12}{5}4^5$$

(v) How many hands contain at least 3 hearts?

- choose 3 hearts $\binom{13}{3}$
- from remaining 49 cards, choose 4 $\binom{49}{4}$

# of hands $= \binom{13}{3}\binom{49}{4}$

(iii)

Ex a) How many 5-arrangements of $\{0,1,\ldots,9\}$ are there (w/ no repetition)?

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$$

b) How many such arrangements are there in which 5 and 6 are not adjacent?

↳ we count the # of arrangements where 5 and 6 are adjacent.

First: those where "56" appears

↳ can think of such an arrangement as a sequence of length 4

__ 56 __ __

where one of the spaces is "56"

to form:

- choose space where "56" appears (4)
- choose digits for remaining three spaces

$$8 \cdot 7 \cdot 6$$

So # of sequences w/ 56 is $4 \cdot 8 \cdot 7 \cdot 6$.

$= 1344$

(iv)

Second: by symmetry, # of arrangements where "65" appears is also 1344

so # of arrangements where 5, 6 non-adjacent is:

$$30,240 - 1344 - 1344$$
$$= 27,552$$ ✓

ex Prove the following identity by counting in two ways:

$$\binom{n}{k} - \binom{n-2}{k} = 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$$

Sol'n: Let S be the set of k-element subsets of $[n] = \{1, 2, \ldots, n\}$ containing either 1 or 2 (or both)

count |S| as

    # of sets containing 1 + not 2
  + # of sets containing 2 + not 1
  + # of sets containing both 1 and 2