Combinatorics ①

↳ the study of counting finite sets
↳ EZ right? no.

Notation: if A is a finite set, $|A|$ denotes the # of el'ts in A

e.g. $|\{*, \heartsuit, \Delta\}| = 3$

Def'n A partition of a finite set A is a collection of pairwise disjoint subsets $\{A_1, \ldots, A_k\}$ of A s.t. $\bigcup_{i \le k} A_i = A$

↳ now we allow some pieces $A_i = \emptyset$ possibly otherwise same def'n as before.

e.g. if $A_1 = \{*, \Delta\}$ $A_2 = \emptyset$, $A_3 = \{\heartsuit\}$ then $\{A_1, A_2, A_3\}$ is a partition of $A = \{*, \heartsuit, \Delta\}$

Principle (Rule of Sum) if $\{A_1, \ldots, A_k\}$ is a partition of a finite set A then:

$$|A| = \sum_{i=1}^{k} |A_i|$$
$$= |A_1| + |A_2| + \cdots + |A_k|$$

Pf: obvious

Ex: Bob's and Rob's are the only two restaurants in town
Bob's offers 9 dinner entrees
Rob's offers 12
How many entree choices are there total for a local who's eating out?

A: $\{\text{all entrees}\} = \{\text{Bob's entrees}\} \cup \{\text{Rob's entrees}\}$

and this is a partition

Hence $|\{\text{all entrees}\}|$
$= |\{\text{Bob's}\}| + |\{\text{Rob's}\}|$

$= 9 + 12 = 21$

dumb but point is: given info about partition pieces, conclude something about entire set.

Principle (Rule of product)

if the elements of a finite set A are formed by making a sequence of k choices s.t.

1. The ith choice can be made in r_i-many ways
2. each element is uniquely formed by such a sequence of choices

then

$$|A| = r_1 r_2 \cdots r_k = \prod_{i=1}^{k} r_i$$

Pf: not illuminating

Ex: - You decide on Rob's for dinner
- w/ each of the 12 entree choices comes a choice of side (8 choices of sides) and drink (6 choices of drink)

How many possible combos of entree, side, drink are there?

A: each combo formed by making seq of 3 choices

Hence # of combos is

$$12 \cdot 8 \cdot 6 = 12 \cdot 48 = (10+2)(40+8)$$
$$= 10 \cdot 40 + 10 \cdot 8 + 2 \cdot 40 + 2 \cdot 8$$
$$= 400 + 80 + 80 + 16$$
$$= 576$$

e.g. "words" like SLE and ZN are included

Ex. How many strings of letters ("words") of length 4 or less can be formed using the English alphabet?

Soln:
- Let A be the set of such strings
- We can partition A as
$$A = A_1 \cup A_2 \cup A_3 \cup A_4$$
where A_i = set of strings of length i, $1 \le i \le 4$.

By rule of sum
$$|A| = |A_1| + |A_2| + |A_3| + |A_4|$$

By rule of product
$$|A_1| = 26$$
$$|A_2| = 26 \cdot 26$$
$$|A_3| = 26 \cdot 26 \cdot 26$$
$$|A_4| = 26 \cdot 26 \cdot 26 \cdot 26$$

So $|A| = 26 + 26^2 + 26^3 + 26^4$
$$= 475,254$$

(of course, if we only wanted to know remainder of $|A|$ when divided by 12 we could calculate:

$$|A| \equiv 2 + 2^2 + 2^3 + 2^4 \pmod{12}$$
$$\equiv 2 + 4 + 8 + 16 \pmod{12}$$
$$\equiv 30 \equiv 6 \pmod{12}$$)

Permutations and arrangements

Def'n if A is a finite set, a permutation of A is an ordered list of the el'ts of A s.t. every el't appears exactly once

e.g. if $A = \{1,2,3\}$
then 213 and 321 are perms of A
but 2231 and 23 are not.

Prop'n: Fix $n \in \mathbb{N} \cup \{0\}$. If A is a finite set of size n then the # of permutations of A is $n!$
(recall $0! = 1$)

Pf: if $|A| = 0$ then $A = \emptyset$ and there is only one permutation of A (the empty permutation)

- if $|A| = n \geq 1$ then a permutation $a_1 a_2 \cdots a_n$ is formed by making a series of n choices

choose a_1, choose a_2, ..., choose a_n

$\downarrow$ n choices, $\downarrow$ $n-1$ choices, $\downarrow$ 1 choice

- so by ROP, # of permutations of A is

$$n \cdot (n-1) \cdots 1 = n!$$

Ex: How many anagrams are there of the word TOY?

Sol'n: - an anagram of TOY is just a perm of the set $\{T, O, Y\}$. (important that TOY has no repeated letters!)

- e.g. YOT is an anagram.

By prop'n # of anagrams is $3! = 6$

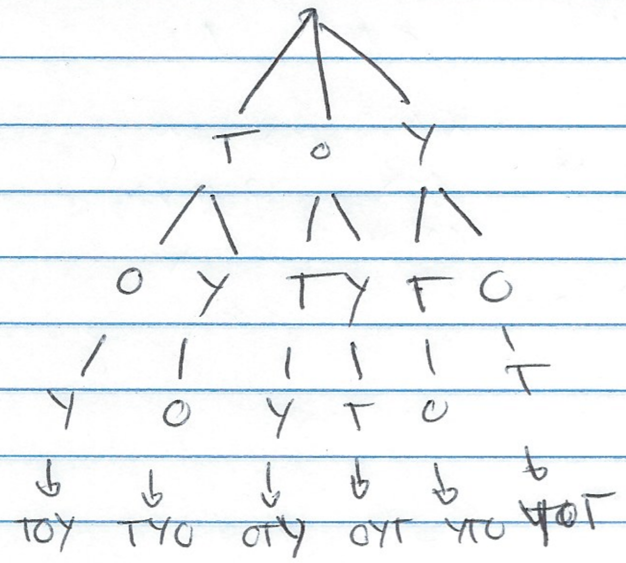
indeed:

TOY YTO

TYO YOT lists them

OTY

OYT



forming anagram as seq of choices:

Def'n Fix $k, n \in \mathbb{N} \cup \{0\}$ with $k \le n$. If A is a set of size n, a k-arrangement of A is an ordered list of k elements of A w/ no repeats

e.g.

- if $A = \{1,2,3,4,5\}$ then 254 and 152 are 3-arrangements of A

- 115 and 29 are not arrangements of A

Prop'n The number of k-arrangements from a set of size n is:

$$n \cdot (n-1) \cdots (n-(k-1))$$

$$= \frac{n!}{(n-k)!}$$

Pf. Let A be a set of size n.
Any k-arrangement
$a_1 a_2 \cdots a_k$
can be formed by making a sequence of k choices:

choose a_1, choose a_2, ..., choose a_k

$\downarrow$ n choices, $\downarrow$ $n-1$, ..., $\downarrow$ $n-(k-1)$

Hence by RoP # of k-arrangements is $n \cdot (n-1) \cdots (n-(k-1))$

$$= \prod_{i=1}^{k-1} n - i$$

$$= n!/(n-k)!$$

Ex. How many strings of English letters of length 3 are there, if no letters are repeated?

Sol'n such a string is just a 3-arrangement of the set:

$$A = \{a, b, \ldots, z\}$$

By prop'n there are $26 \cdot 25 \cdot 24 = 15{,}600$ such strings. ✓

Selections

Def'n Fix $n, k \in \mathbb{N} \cup \{0\}$ with $k \leq n$. If A is a set of size n, a <u>k-selection</u> of A (or <u>k-combination</u>) is a subset of A (i.e. an <u>unordered</u> list of el'ts of A)

e.g. if $A = \{1,2,3,4\}$ then $\{2,3\}$ is a 2-selection of A

<u>Notation</u>: $\binom{n}{k}$ denotes the # of k-selections from a set of size n.

Prop'n For $k \leq n$ we have:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Pf: - Let A be a set of size n and let S be the set of <u>k-arrangements</u> of A.
- we will count $|S|$ in two ways.
- From prev. prop'n we know

$$|S| = \frac{n!}{(n-k)!}$$

- OTOH: an element of S (i.e. a k-arrangement of A) can be formed by making two choices.
  - first choose a k-selection $\{a_1, \dots, a_k\} \subseteq A$.
  - then choose an ordering (i.e. a permutation) of this selection

- there are (by def'n) $\binom{n}{k}$-many ways of making first choice, and then after $k!$-many ways of making the second choice

- Hence by ROP:

$$|S| = k! \binom{n}{k}$$

i.e.
$$\frac{n!}{(n-k)!} = k! \binom{n}{k}$$

so:
$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

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e.g. $\binom{10}{3} = \frac{10!}{7!3!} = \frac{\cancel{10} \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!} \cdot 3 \cdot 2 \cdot 1} = 40$

$\binom{10}{7} = \frac{10!}{3!7!} = \binom{10}{3}$ in general: $\binom{n}{k} = \binom{n}{n-k}$

(i)

Ex: How many ways are there to choose a chief and two henchpersons from a group of 10 people.

Sol'n: - 10 choices for chief
- after chief chosen, $\binom{9}{2}$ choices for henchpeople.

so # of such committees is:

$$10 \times \binom{9}{2}$$

$$= 10 \times \frac{9!}{7!\,2!}$$

$$= 10 \times \frac{9 \cdot 8}{2 \cdot 1} = 360$$

Alt. Sol'n - or we could first select group of 3
- then from these 3 select chief
- so # of possible committees is

$$\binom{10}{3} \cdot 3 = \frac{10!}{3!\,7!} \cdot 3$$

$$= \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} \cdot 3$$

$$= 360, \text{ as before} \checkmark$$

(ii) Countin' Poker Hands

- a deck of cards consists of 52 cards.
- each card has one of 4 possible suits (♡, ♢, ♣, ♠) and one of 13 possible ranks (A, 2, 3, ..., 9, 10, J, Q, K)
- e.g. A♡ and 2♢ are cards
- a poker hand is a selection of 5 cards from a standard deck

Ex ① How many distinct hands are possible?

Sol'n: $\binom{52}{5} = 2,598,960$

② A full house is a hand consisting of three cards of one rank and two cards of another
(e.g. 3♡, 3♢, 3♣, J♢, J♠)

How many distinct full house hands are possible?

Sol'n: - Pick two ranks $\binom{13}{2}$
- From these, pick the three card rank $\binom{2}{1}$
- pick three cards from the three card rank $\binom{4}{3}$

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- Pick two cards from the two card rank $\binom{4}{2}$

so: # of full house hands is
$$\binom{13}{2}\binom{2}{1}\binom{4}{3}\binom{4}{2} = 3,744$$

(3) A 3-of-a-kind is a 5 card hand in which three cards are from a single rank and two cards from two other distinct ranks

e.g. three 4's and a J and a Q

Q: How many 3-of-a-kind hands are there?

Sol'n:
- Pick the three card rank $\binom{13}{1}$
- From this rank, pick 3 cards $\binom{4}{3}$
- Pick remaining two ranks $\binom{12}{2}$
- From the first of these, pick a card $\binom{4}{1}$
- and one from the second $\binom{4}{1}$

so: # of 3-of-a-kinds is:
$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$$
$$= 54,912$$

(iv)

<u>Alt Soln</u>: - Pick the three ranks $\binom{13}{3}$
- From them, pick the 3-card rank $\binom{3}{1}$
- From this rank, pick 3 cards $\binom{4}{3}$
- For the other two ranks, pick cards $\binom{4}{1}\binom{4}{1}$

But: $\binom{13}{3}\binom{3}{1}\binom{4}{3}\binom{4}{1}\binom{4}{1} = 54{,}912$ fear

<u>Binary Sequences</u>

- a <u>binary sequence</u> (of length n) is an ordered sequence of 0's and 1's (of length n)

- e.g. $s = 011$ is a binary sequence of length 3

- we denote the set of all binary sequences of length n by P_n.

<u>Ex</u> (1) how many sequences $s \in P_n$ have <u>at least</u> two 1's?

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Sol'n: - easier to count # of sequences with zero 1's or one 1 then subtract from total # of sequences

- so first we compute $|P_n|$.
- each $s \in P_n$ formed by making n choices:

$$\underbrace{\underline{0/1}\ \underline{0/1}\ \underline{0/1}\ \cdots\ \underline{0/1}}_{n}$$

2 choices each

hence: # of total sequences is
$2 \cdot 2 \cdot \cdots 2 \cdot = 2^n = |P_n|$ ✓

Now: # of seq's w/ zero 1's is 1
(just all 0 - sequence)
# of seq's w/ a single 1 is
$n = \binom{n}{1}$

Hence # of sequences w/ at least two 1's is
$2^n - n - 1$ ✓

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Theorem Fix $n \in \mathbb{N} \cup \{0\}$.
Then

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$$

Pf: - if $n = 0$ then

$$2^0 = 1 = \binom{n}{0}$$

- So suppose $n \geq 1$
- we proved above $|P_n| = 2^n$
- we can partition P_n as follows:

$$P_n = S_0 \cup S_1 \cup \cdots \cup S_n$$

- where S_k = set of sequences w/ exactly k 1's.

Observe: $|S_k| = \binom{n}{k}$

# of ways to pick k positions where 1's appear.

Hence
$$2^n = |P_n| = |S_0| + |S_1| + \cdots + |S_n| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} \checkmark$$

Selections w/ repetition

General Q: How many ways to select n objects from k types of objects, if repetition is allowed?

Ex: – Dee's donuts sells 4 types of donuts.
– You want to buy a dozen.
– How many different ways of doing this?

Sol'n:

(1)

- Imagine putting down 3 "spacers"

$\swarrow$ type 1 $\swarrow$ type 2 $\swarrow$ type 3 $\swarrow$ type 4

o o o | o o | o o o o o | o

- put a o to left of first spacer for every donut of type I,
- put a o between first and second spacer for every donut of type 2,
- etc.

- Can view "donut + spacer" diagram as a 01-sequence w/ 12 0's (for dozen donuts) and 3 1's (for separating 4 types)

So: 0001001000001 0

corresponds to an order of

3 type 1 donuts
2 type 2 "
5 type 3 "
1 type 4 "

- Conversely, any such sequence (12 0's, 3 1's) corresponds to a selection of donuts e.g.

101 0000000 1 0000

corresponds to a selection of

0	tp1	donuts
1	tp2	donuts
7	tp3	"
4	tp4	"

<u>Hence</u>: # of ways to make a selection
= # of ways to sprinkle 3 1's amongst 12 0's
= # of 01-seqs of length 15 w/ 3 1's
= $\binom{15}{3} = 455$

Same reasoning in general proves:

<u>Theorem</u> The # of ways to make a selection of n objects from k types w/ repetition allowed is

$$\binom{n+(k-1)}{k-1}$$

↳ $k-1$ because only need $k-1$ spacers to separate k types of objects

(iii) so $2+3+2=7$ is diff than $3+2+2=7$ a solution is a triple (x,y,z)

ex: How many non-negative solutions $x,y,z \in \mathbb{N} \cup \{0\}$ are there to the equation

$$x+y+z=7\,?$$

Sol'n: can think of a sol'n as a partition of 7 o's into 3 piles:

$$\overset{x}{oo} \mid \overset{y}{ooo} \mid \overset{z}{oo}$$

would correspond to $(x,y,z)=(2,3,2)$

$$\mid ooooo \mid oo$$

would correspond to $(x,y,z)=(0,5,2)$

Hence # of sol'ns is just

$$\binom{7+2}{2} = \binom{9}{2} = 36$$

ex: Suppose we roll n (indistinguishable) 6-sided dice.

① How many distinct outcomes are possible?

Sol'n: each of the n dice can roll into 6 possible "types"

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1	2	3	4	5	6
oo	oo		oooo		ooo

hence # of possible outcomes is:

$$\binom{n+(6-1)}{6-1} = \binom{n+5}{5}$$

② Assume $n \geq 12$. How many outcomes are possible if every value 1, 2, 3, 4, 5, 6 is rolled at least twice?

Sol'n: - Put 2 dice in each of the 6 categories

- remaining $n-12$ dice can now be rolled arbitrarily

- there are:

$$\binom{(n-12)+(6-1)}{(6-1)} = \binom{n-7}{5}$$

ways to do this ✓

Ex How many anagrams of the word

LIMITING

are there?

(the three I's being indistinguishable)

(v)

Sol'n : Two approaches

ⓐ - distinguish the I's w/ subscripts I_0, I_1, I_2

- number of anagrams w/ distinguished I's is just $8!$

- For each anagram w/ I's distinguished there are $3!$ equivalent anagrams when the I's are not distinguished

hence # of anagrams is

$$\frac{8!}{3!}$$

ⓑ Alternately can think of anagram as being formed in two stages

_ _ _ _ _ _ _ _

(i) Pick 3 positions for the I's $\binom{8}{3}$

(ii) for remaining 5 positions pick an ordering of LMTNG $5!$

So: # of anagrams is

$$\binom{8}{3} \cdot 5! = \frac{8!}{5!\,3!}\,5! = \frac{8!}{3!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$

6600
+ 6·20

=

6720

(vi)

Counting in two ways

idea: if we can think of two ways to count the elements of a set, the expression must be equal.

Thm (Pascal's Identity)
Fix $n, k \in \mathbb{N}$ with $k \leq n$, then:
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pf: - Let S be the set of k element subsets of $[n] = \{1, 2, \ldots, n\}$
- then $|S| = \binom{n}{k}$

OTOH we can partition S into S_1 and T, where
S_1 = subsets of $[n]$ of size k that contain 1, and
T = subsets of $[n]$ of size k that do not contain 1

Then: $|S| = |S_1| + |T|$

- subsets in S_1 are formed by selecting $k-1$ el'ts from $\{2, \ldots, n\}$

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so: $|S_1| = \binom{n-1}{k-1}$

- subsets in T are formed by selecting k elements from $\{2, \ldots, n\}$
  ↪ so $|T| = \binom{n-1}{k}$

- So: $|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$
  i.e. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ ✓