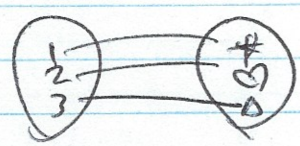
Infinity

Cardinality

- we would say {*, ♡, Δ} has 3 elements, or is of size 3.
- why? By counting

{*, ♡, Δ}
1 2 3

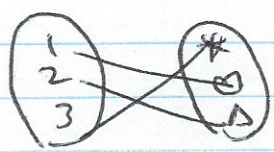
- in "counting" we are implicitly defining a bijection between {1,2,3} and {*, ♡, Δ}



- we could've counted differently

{*, ♡, Δ}
3 1 2

i.e.



↳ generalizing this idea: we will say two sets are of the same size iff there is a bijection between them

Def'n We say that two sets A, B have the same cardinality iff $\exists f: A \to B$ a bijection.
In this case we write $A \sim B$.

Note: - in set theory courses, for every set A one defines the cardinal number $|A|$ of A.
- can then prove: $A \sim B$ iff $|A| = |B|$
(e.g. $|\{*, \alpha, \Delta\}| = |\{\circ, \square, \Diamond\}| = 3$)
- defining cardinal numbers beyond our scope:
For us:
$|A| = |B|$ just means
$A \sim B$ i.e.
$\exists f: A \to B$ a bijection.

Properties of $\sim$:

① For any set A, $id_A: A \to A$ is a bijection. (Why?) Hence $A \sim A$, i.e. $\sim$ is reflexive.

② If $f: A \to B$ is a bijection then f is invertible and $f^{-1}: B \to A$ is a bijection too.
Hence $A \sim B$ implies $B \sim A$.
i.e. $\sim$ is symmetric.

(3) on HW you guys showed: if $f: A \to B$ and $g: B \to C$ are bijections then $g \circ f: A \to C$ is a bijection. Hence if $A \sim B$ and $B \sim C$ then $A \sim C$, i.e. $\sim$ is <u>transitive</u>.

↳ $\sim$ is an equivalence relation on sets!

↳ $\sim$ is most interesting when sets being compared are infinite.

<u>ex's:</u> (1) Let $-\mathbb{N}$ denote $\{\ldots, -3, -2, -1\}$. define

$$f: \mathbb{N} \to -\mathbb{N}$$
$$f(n) = -n$$

<u>Conclude</u>: f is a bijection. hence $\mathbb{N} \sim -\mathbb{N}$.

(2) Last time we proved that

$$f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$

defines a bijection $f: \mathbb{Z} \to \mathbb{N}$.

hence $\mathbb{Z} \sim \mathbb{N}$.

Def'n Let A, B be sets

① We write $A \lesssim B$ (or $|A| \le |B|$) iff $\exists$ an injection $f: A \to B$

② We write $A \gtrsim B$ (or $|A| \ge |B|$) iff $\exists$ a surjection $f: A \to B$.

↳ we'll write $A < B$ to mean $A \lesssim B$ but $A \not\approx B$.

NOTE $A \gtrsim B$ is not "reverse of" $A \lesssim B$, i.e. it is not asserting there is an injection from B to A. But this follows:

Theorem For all sets A, B we have

$$A \lesssim B \text{ iff } B \gtrsim A$$

i.e. $\exists f: A \to B$ an injection iff $\exists g: B \to A$ a surjection.

Pf – proof in general requires axiom of choice

– we'll do a "proof by two examples" (i.e. not a proof at all) to show idea.

($\Rightarrow$) Sps $A \lesssim B$ i.e. $\exists f: A \to B$ an injection.

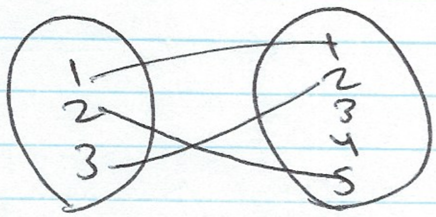
WTS: ~~$\exists g$~~ $B \gtrsim A$ i.e. $\exists g: B \to A$ a surjection

Consider

$f : [3] \to [5]$

defined by $f(1) = 1$
$f(2) = 5$
$f(3) = 2$

<u>Observe</u>: f is an injection

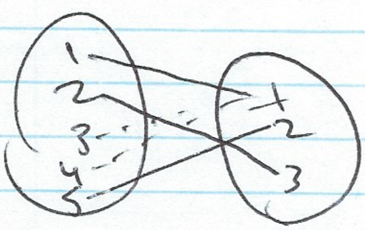


<u>idea</u>: to get a surjection $g : [5] \to [3]$ take "reverse of f" where we can then map anything left over to something arbitrary.

define $g : [5] \to [3]$
by $g(1) = 1$
$g(2) = 3$
$g(5) = 2$

} - "reverses" f
- already from this info know g is surjective

then let $g(3) = g(4) = 1$



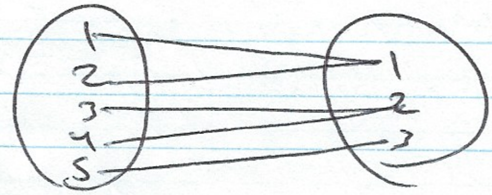
g a surjection!
g a function cause f was injective ✓

↳ this idea works in general.

(⇐) – Sps $B \gtrsim A$
i.e. $\exists\, g: B \to A$ a surjection
– want to construct an injection $f: A \to B$.

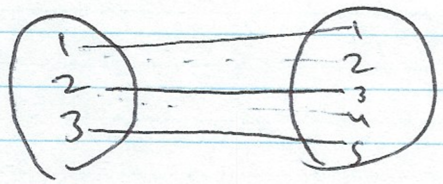
<u>idea</u>: for every $x \in A$, pick some element $y_x \in \mathrm{PreIm}_g(\{x\})$ (which is $\neq \emptyset$ because g is surj) then let $f(x) = y_x$

<u>ex</u> Suppose $g: [5] \to [3]$ is the function
$g(1) = 1$ $g(3) = 2$ $g(5) = 3$
$g(2) = 1$ $g(4) = 2$



g

observe: g is a surjection



"reverses g" w/o repeats → injective cause g was surjective

define $f: [3] \to [5]$ by
$f(1) = 1$
$f(2) = 3$
$f(3) = 5$

→ same idea works in general.

Properties of $\lesssim$ and $\gtrsim$.

Sps A, B, C are sets

① $A \lesssim A$ and $A \gtrsim A$ since $id_A : A \to A$ is both an injection and surjection. Hence $\lesssim$ and $\gtrsim$ are reflexive.

② if $A \lesssim B$ and $B \lesssim C$ then $\exists f : A \to B$ and $g : B \to C$ injections. Then $g \circ f : A \to C$ is an injection (by HW) so $A \lesssim C$

Sim'ly $A \gtrsim B$ and $B \gtrsim C \Rightarrow A \gtrsim C$

→ so $\lesssim$ and $\gtrsim$ are transitive.

③ Are they antisymmetric? Not literally: if $A \lesssim B$ and $B \lesssim A$ then not necessarily true $A = B$

e.g. $A = \{1, 2, 3\}$
$B = \{*, \triangleleft, \triangle\}$

But we'll show in such a case that $A \sim B$! (CSB theorem)

Cardinality notation: just like $|A| = |B|$ means $A \sim B$
we'll write $|A| \leq |B|$ for $A \lesssim B$
and $|A| \geq |B|$ for $A \gtrsim B$.

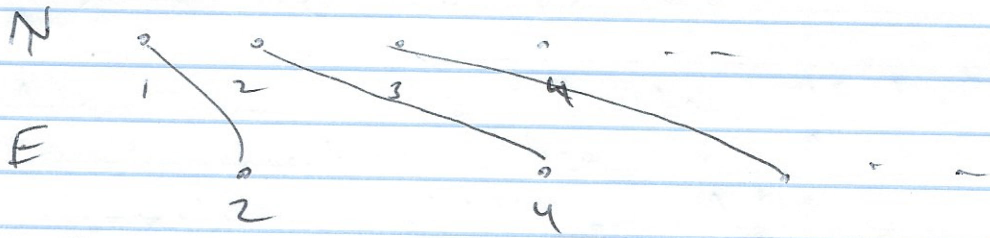
↓ but first:

Some paradoxes of infinity

$A \sim B$ when A, B infinite can be counterintuitive!

1. Let $E = \{2, 4, 6, \ldots\}$
   then $N \sim E$ ("there are as many even numbers as whole numbers")

   pf: $f: N \to E$
   defined by $f(n) = 2n$ is a bijection

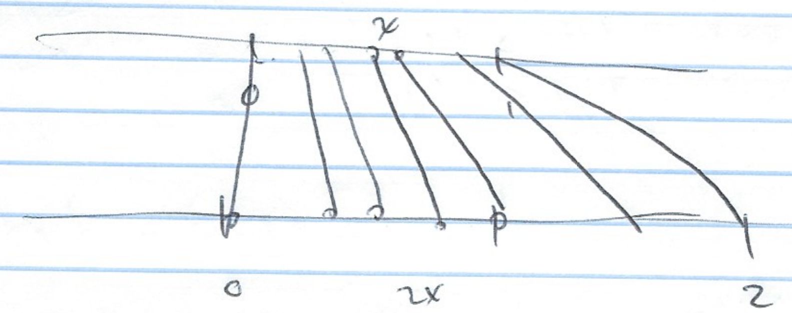


2. Let $S = \{1, 4, 9, 16, \ldots\}$ = set of squares
   then
   $N \sim S$ ("there are as many squares as whole numbers")

   pf: $f: N \to S$
   $f(n) = n^2$ is a bijection

3. Let $[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$
   $[0,2] = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\}$
   then $[0,1] \sim [0,2]$ ("there are as many #s between 0 and 1 as between 0 and 2")

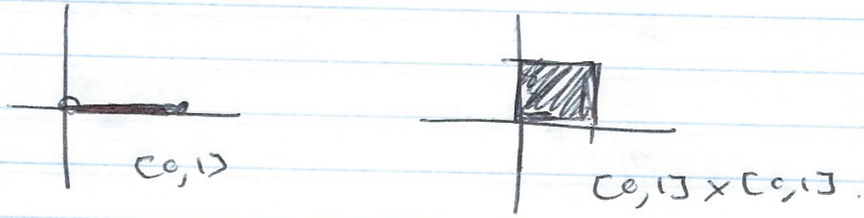
   pf: $f: [0,1] \to [0,2]$
   $f(x) = 2x$ is a bijection



④ In fact "the side is as large as the square"

i.e. there is a bijection $f: [0,1] \to [0,1] \times [0,1]$

(pf is beyond our scope)



Moral: infinite sets are wild!

<u>Def'n</u> - a set X is <u>finite</u> iff $X = \emptyset$ or $\exists n \in \mathbb{N}$ and a bijection $f: [n] \to X$

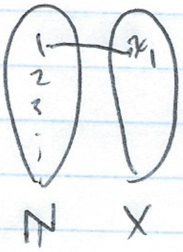
- X is <u>infinite</u> iff it is not finite.

↪ i.e. $X \neq \emptyset$ and $\forall n \in \mathbb{N}$ there is no bijection $f: [n] \to X$

↪ $\mathbb{N}$ has the "smallest" size an infinite can have, in the following sense:

<u>Theorem</u>: if X is infinite then $\mathbb{N} \precsim X$.

<u>Pf</u>: we define an injection $f: \mathbb{N} \to X$ inductively



(BC) since $X \neq \emptyset$ there is $x_1 \in X$.
Let $f_1 = \{(1, x_1)\}$

(IH) Sps at stage n we have defined a "partial injection" f_n

i.e. $f_n = \{(1,x_1), (2,x_2), \ldots, (n,x_n)\}$

s.t. $\forall i,j \leq n$ if $i \neq j$ then $x_i \neq x_j$

(IS) then since X is not finite
$f_n : [n] \to X$ is not a bijection
Since it is an injection, this means
f_n is not a surjection
i.e. $\exists\, x_{n+1} \notin \{x_1, \dots, x_n\}$
$x_{n+1} \in X$.

Let $f_{n+1} = \{(1,x_1), \dots, (n,x_n), (n+1, x_{n+1})\}$

by induction we can define, $\forall n \in \mathbb{N}$,
an injection $f_n : [n] \to X$
s.t. $n \le m \Rightarrow f_n \subseteq f_m$.

Let $f = \bigcup_{n \in \mathbb{N}} f_n = \{(1,x_1), (2,x_2), \dots\}$

then $f : \mathbb{N} \to X$ is an injection. ✓
i.e. $\mathbb{N} \preceq X$.

— Thm says $\mathbb{N}$ is as small as possible for an infinite set: even sets that may appear smaller (e.g. E, O, etc) are not!

— OTOH many sets which seem larger than $\mathbb{N}$ are not as well

Def'n A set X is <u>countable</u> iff $\mathbb{N} \sim X$.

Ex's: ① $\mathbb{Z}$ is countable
Pf: we showed that
$f : \mathbb{Z} \to \mathbb{N}$

defined by

$$f(n) = \begin{cases} 2n & n > 0 \\ 2(-n)+1 & n \leq 0 \end{cases}$$

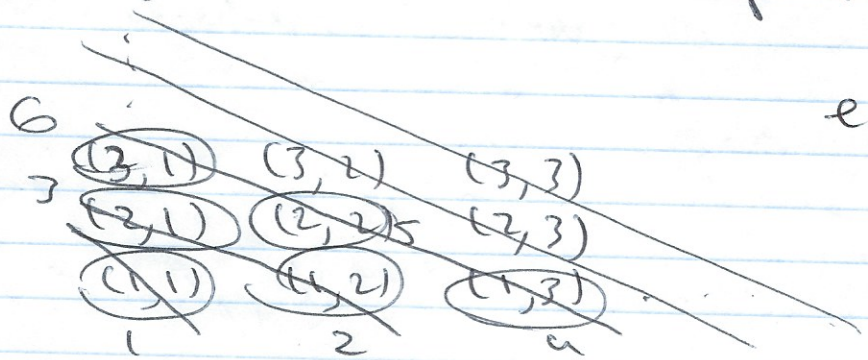
is a bijection ✓

(2) $N \times N$ is countable

Pf: we need to produce a bijection $f: N \to N \times N$.

possible to do so explicitly but we just draw the picture:

$N \times N$:



e.g.
$f(1) = (1,1)$
$f(2) = (1,2)$
$f(3) = (2,1)$
$f(4) = (1,3)$
⋮

etc.

Continuing in this way (counting along diagonals) we get a bijection

$$f: N \to N \times N$$

Sometimes hard to show $A \sim B$ directly, but easy to show $A \precsim B$ and $B \precsim A$.

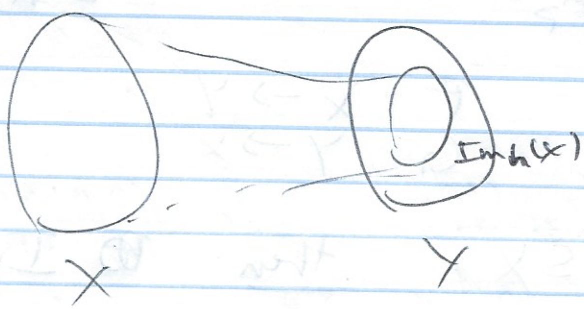
Turns out, this is good enough!

Theorem (Cantor-Schroeder-Bernstein)
For any sets A, B, if $A \lesssim B$ and $B \lesssim A$ then $A \sim B$.

Pf. Sps we have injections
$f: A \to B$
$g: B \to A$

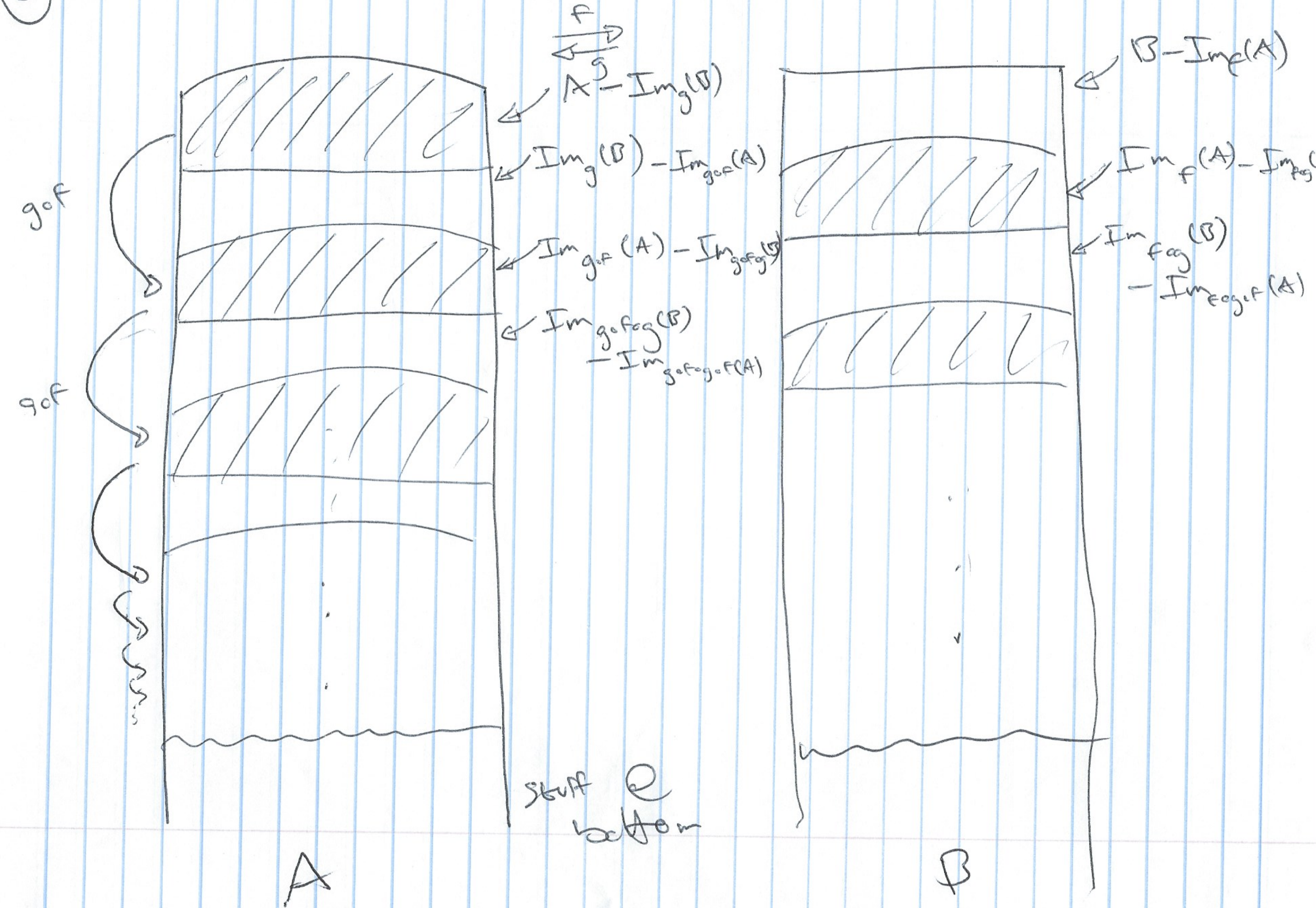
we need to construct a bijection $F: A \to B$, i.e. $A \sim B$

side note: in general if $h: X \to Y$ is an injection then h is a bijection of X with $Im_h(X) \subseteq Y$ i.e. $X \sim Im_h(X)$



In our case we have $A \sim Im_f(A) \subseteq B$
$B \sim Im_g(B) \subseteq A$

hence to prove $A \sim B$ it is enough to prove $A \sim Im_g(B)$ since we know $Im_g(B) \sim B$.



f
g
A - Im_g(B)
Im_g(B) - Im_{g∘f}(A)
Im_{g∘f}(A) - Im_{g∘f∘g}(B)
Im_{g∘f∘g}(B) - Im_{g∘f∘g∘f}(A)
g∘f
g∘f
stuff @ bottom
A
B - Im_f(A)
Im_f(A) - Im_{f∘g}
Im_{f∘g}(B) - Im_{f∘g∘f}(A)
B

"believe me..."

<u>Observe</u>: $g \circ f$ gives a bijection of each shaded piece with "the next one down"

hence if we define

$$F: A \to Im_g(B)$$

by

$$F(x) = \begin{cases} g \circ f(x) & \text{if } x \text{ in shaded region} \\ x & \text{otherwise} \end{cases}$$

then F is a bijection of A with $Im_g(B)$

hence $A \sim Im_g(B)$
and since $Im_g(B) \sim B$
we have $A \sim B$ ✓

↳ CSB then says: while $\lesssim$ is not literally antisymmetric, is a.s. up to $\sim$

$$A \lesssim B \text{ and } B \lesssim A \implies A \sim B$$

↳ Since we knew $A \lesssim B$ iff $B \gtrsim A$
CSB also gives
$B \gtrsim A$ and $A \gtrsim B$ then $A \sim B$

Next goal: use CSB to prove that $\mathbb{Q}$ is countable!

First need:

<u>Theorem</u> if A, B are ctbl sets, then $A \times B$ is ctbl.

Pf: Sps we have bijections
$$f: \mathbb{N} \to A$$
$$g: \mathbb{N} \to B$$

- We know from before: $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$
- so if we can show $\mathbb{N} \times \mathbb{N} \sim A \times B$, we'll be done
- consider
$$F: \mathbb{N} \times \mathbb{N} \to A \times B$$
defined by $F(n,m) = (f(n), g(m))$

<u>Claim</u>: F is a bijection

<u>Pf</u>: - Fix $(a,b) \in A \times B$. Since f, g surjective, $\exists\, n, m \in \mathbb{N}$ s.t. $f(n) = a$, $g(m) = b$
 - Hence $F(n,m) = (f(n), g(m)) = (a,b)$
 - Hence F is surjective

 - if $F(n,m) = F(n', m')$
 then $(f(n), g(m)) = (f(n'), g(m'))$
 - hence $f(n) = f(n')$, $g(m) = g(m')$ $\Rightarrow$ $m = m'$, $n = n'$ since f, g injective

— So $(n, m) = (n', m')$
— So F is injective ✓

Hence $\mathbb{N} \times \mathbb{N} \sim A \times B$
Hence $\mathbb{N} \sim A \times B$ ✓

<u>Theorem</u> $\mathbb{Q}$ is countable, i.e. $\mathbb{N} \sim \mathbb{Q}$

<u>Pf.</u> — $f: \mathbb{N} \to \mathbb{Q}$ defined by $f(n) = n$ is an injection
— hence $\mathbb{N} \precsim \mathbb{Q}$
— so we just need to show $\mathbb{Q} \precsim \mathbb{N}$ (by CSB).

— above we proved $\mathbb{Z} \sim \mathbb{N}$
— so by prev. thm $\mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$
— we also have:

<u>Claim</u>: $\mathbb{Z} \times \mathbb{N} \succsim \mathbb{Q}$.

<u>Pf.</u> — $F: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ defined by $F(m, n) = \frac{m}{n}$ is a surjection
— <u>why</u>, if $q \in \mathbb{Q}$ and $q = m/n$ then $F(m, n) = m/n = q$.
— F is <u>def</u> <u>not</u> injective, e.g. $F(1, 2) = F(3, 6)$, etc.
— but who cares! we've still shown $\mathbb{Z} \times \mathbb{N} \succsim \mathbb{Q}$

— hence ~~[illegible]~~ $\mathbb{Q} \precsim \mathbb{Z} \times \mathbb{N} \sim \mathbb{N}$
— hence $\mathbb{Q} \precsim \mathbb{N}$
— but then CSB gives $\mathbb{Q} \sim \mathbb{N}$!

↪ so $\mathbb{N}$ is "small" in that $\mathbb{N} \lesssim X$ for every infinite set X
↪ OTOH $\mathbb{N}$ is "large": many sets X which seem larger than $\mathbb{N}$ actually have $X \sim \mathbb{N}$.

Q: Are there infinite sets X s.t. $\mathbb{N} \not\sim X$?

A: Yes!

<u>Theorem</u> (Cantor) $\mathbb{N} < P(\mathbb{N})$. That is: $\mathbb{N} \lesssim P(\mathbb{N})$ but $\mathbb{N} \not\sim P(\mathbb{N})$

<u>Pf</u>: - $\mathbb{N} \lesssim P(\mathbb{N})$ is true since $P(\mathbb{N})$ is infinite
- need show: $\mathbb{N} \not\sim P(\mathbb{N})$
- will actually show:

<u>Claim</u>: Let $f: \mathbb{N} \to P(\mathbb{N})$ be a function. Then f is <u>not</u> a surjection.

<u>Pf</u>: a magic trick

- Let $T = \{n \in \mathbb{N} \mid n \notin f(n)\}$

<u>illustration</u>: e.g. if

$f(1) = \{1, 7, 10\}$
$f(2) = \{1, 3, 5, 7, 9, \ldots\}$

$f(3) = \emptyset$
$f(4) = \{2, 4, 6, 8, \ldots$

⋮

then: $1 \notin T$ because $1 \in f(1)$
$2 \in T$ because $2 \notin f(2)$
$3 \in T$ because $3 \notin f(3)$
$4 \notin T$ because $4 \in f(4)$

⋮

etc.

So $T = \{2, 3, \ldots\}$

Then: $(\forall n \in \mathbb{N})\ f(n) \neq T$

Pf: - Fix $n \in \mathbb{N}$.

Two cases: - if $n \in T$, then $n \notin f(n)$ by def'n of T
- hence $T \neq f(n)$ $\left(\begin{smallmatrix} n \in T \\ n \notin f(n) \end{smallmatrix}\right)$

- if $n \notin T$ then $n \in f(n)$
- hence again $T \neq f(n)$ $\left(\begin{smallmatrix} n \notin T \\ n \in f(n) \end{smallmatrix}\right)$

↳ Since n was arbitrary, $f(n) \neq T$ always.

But now the claim follows: $T \notin \operatorname{Im} f$
so f is not a surjection ✓

Since f was arbitrary, there is no surjection $f: \mathbb{N} \to P(\mathbb{N})$, hence no bijection
hence $\mathbb{N} \not\approx P(\mathbb{N})$ ✓

(i)

Same proof works in general.

<u>Theorem</u> For any set A, there is no surjection $f: A \to P(A)$

<u>Pf</u>: Fix $f: A \to P(A)$
Let $T = \{a \in A \mid a \notin f(a)\}$
<u>Then</u> (same arg as before)
$\forall a \in A \quad f(a) \neq T$. ✓

↳ Since $g: A \to P(A)$ defined by $g(a) = \{a\}$ is always an injection we always have $A \leq P(A)$
↳ Hence theorem gives $A < P(A)$

It follows that there are ∞-many levels of ∞:

$$\mathbb{N} < P(\mathbb{N}) < P(P(\mathbb{N})) < \ldots$$

<u>Def'n</u> If X is infinite and $\mathbb{N} \not\approx X$ then we say X is <u>uncountable</u>.

↳ Thm above: $P(\mathbb{N})$ is uncountable
↳ What about $\mathbb{R}$?

<u>Theorem</u> $\mathbb{R}$ is uncountable.

<u>Pf</u>: We'll actually show that a particular <u>subset</u> of $\mathbb{R}$ is unctbl.

- Let $R = \{ x \in (0,1) \mid$ decimal expansion of x consists only of 0's and 1's $\}$

e.g.

$.1011001\ldots \in R$

$.101010\ldots \in R$

$.23110\ldots \notin R$

$1.1011\ldots \notin R$.

For $x \in R$, define x_n to be nth digit of decimal expansion of x

so: $x = .x_1 x_2 x_3 \ldots$

eg. if $x = .0110\ldots$
then $x_1 = 0$
$x_2 = 1$
$x_3 = 1$.

Claim: if $f: \mathbb{N} \to R$ is any function, then f is not a surjection.

Pf: Fix $f: \mathbb{N} \to R$.
We define $r \in R$ as follows

- $r_n = 0$ if $f(n)_n = 1$
- $r_n = 1$ if $f(n)_n = 0$

Picture: Sps

$f(1) = .\textcircled{1}100101\ldots$
$f(2) = .0\textcircled{1}00111 0\ldots$
$f(3) = .11\textcircled{0}1001\ldots$
$f(4) = .101\textcircled{0}11\ldots$

$r_1 = 1 \quad r_2 = 1 \quad r_3 = 0 \quad r_4 = 0$

so $r = .0011\ldots$

Observe:

$(\forall n \in \mathbb{N})\ r \neq f(n)$

Why: $r_n \neq f(n)_n$
i.e. r and $f(n)$ differ in the nth place!

hence f is not a surjection ✓

↳ hence $\mathbb{N} \not\approx R$
so $\mathbb{N} < R$ ← identity injection
and since $R \lesssim \mathbb{R}$
we have $\mathbb{N} < \mathbb{R}$ ✓

Sets of Functions

Consider the set F of functions $f: \mathbb{N} \to \{0,1\}$

$F = \{ f \subseteq \mathbb{N} \times \{0,1\} \mid f: \mathbb{N} \to \{0,1\}$ is a function$\}$

- we can think of a given $f \in F$ as an infinite 01-sequence.

e.g. if $f: \mathbb{N} \to \{0,1\}$

$$f(1) = 0$$
$$f(2) = 0$$
$$f(3) = 1$$
$$f(4) = 1$$
$$f(5) = 0$$
$$\vdots$$

Can picture f like this:

$$f = 0\,0\,1\,1\,0 \ldots$$

So e.g. if I write

$$g = 1\,0\,1\,0\,1\,0 \ldots$$

then

$$g(1) = 1$$
$$g(2) = 0$$
$$g(3) = 1$$
$$\vdots$$

etc.

Theorem F is uncountable!

Pf: Diagonalize.

Claim: if $H: \mathbb{N} \to F$ is a function then H is not a surjection

Pf: define a function
$f: N \to \{0,1\}$
as follows:

$$f(n) = \begin{cases} 1 & \text{if } H(n)(n) = 0 \\ 0 & \text{if } H(n)(n) = 1 \end{cases}$$

then $(\forall n \in N)$ $f \neq H(n)$
Why: $f(n) \neq H(n)(n)$ by def'n!

So f, $H(n)$ differ in the nth place!

e.g. if
$H(1) = \textcircled{0}\,1\,0\,1$ ← $h(1)(1)$
$H(2) = 0\,\textcircled{0}\,1\,1\,0\,1\,0\,1\,1 \ldots$ ← $h(2)(2)$
$H(3) = 1\,1\,\textcircled{1}\,1\,1\,1 \ldots$ ← $h(3)(3)$
$H(4) = 0\,0\,0\,\textcircled{1}\,1\,1 \ldots$ ← $h(4)(4)$

then $f = 1\,1\,0\,0 \ldots$

↓ the claim follows ✓