# 21-127 Concepts of Math

Garrett Ervin
gervin@andrew.cmu.edu
7128 Wean Hall

Office hours (for now): M 10:30 – 12:00
F 2:30 – 4:00

course website: canvas.cmu.edu/courses/8974

## Grading:

HW = 30% (1/week)
Quizzes = 10% (1/week in recitation)
Midterm 1 = 15%
Midterm 2 = 15%
Final = 30%

Textbook: Sullivan (download on Canvas)

HW, solutions, etc. all posted on Canvas.

## Overview

- Class is an intro to writing proofs

Math is broad!

- no single area of focus: will cover basic set theory, logic, number theory, combinatorics, and topology.

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↳ what is "doing math?"
↳ not just calculating!

**Roughly**: math is investigation of mathematical **objects** or **concepts** (e.g. integers, right triangles, manifolds) by way of proving the truth/falsity of **mathematical statements** (e.g. "every diagonal matrix is invertible") about these objects.

- mathematical concepts are described by precise **definitions**

e.g.
**Def'n**: a prime number is a positive integer $p$, such that if $n$ is a positive integer that divides $p$, then either $n=1$ or $n=p$.

**Not def'ns**: - "a **line** is a flowing point."
- "a **point** is a place without extension" - Emerson

↳ suggestive but not precise.

- Mathematical statements (or propositions) are declarative sentences (concerning mathematical objects) that are either true or false.

- e.g.

Euclid: "There are more primes than found in any list of primes."

Prop'n 1: There are infinitely many prime numbers

↳ is true or false: either there are infinitely many primes, or not. (in fact, there are)

↳ establishing truth requires a proof.

- roughly: a sequence of logical deductions from axioms or previously proved statements whose conclusion is the prop'n in question

- many methods of proof: one is by contradiction.

Proof of prop'n 1: Suppose toward a contradiction that there are only finitely many primes (i.e. that prop'n 1 is false)

- then we can list them as
  $p_1, p_2, \ldots, p_n$
- consider the integer
  $$N = p_1 \cdot p_2 \cdots p_n + 1$$
  formed by multiplying the primes and adding 1.
- Observe: if we divide N by any of the primes $p_1, \ldots, p_n$ we leave a remainder of 1
- Hence N is not divisible by any of $p_1, \ldots, p_n$
- Thus N must itself be prime, or there is another prime p not among $p_1, \ldots, p_n$
- in either case there is another prime not among $p_1, \ldots, p_n$ a contradiction, as we assumed these were all primes
- Hence our assumption was false.
- Hence there are infinitely many primes.

# Sets

- A set is a collection of objects (often defined by a common property)

- Cantor: "By a 'set' we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought."

- this is an informal def'n (and in fact contradictory)

- formal def'n of set beyond scope of course

- our approach: we will write down several fundamental sets that we "take for granted" and then give formal def'ns of certain operations that allow us to build new sets from old ones.

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- sets are enclosed by curly brackets $\{ \cdots \}$

- objects in a set are called elements

- $\in$ means "is an element of"
  $\notin$ means "is not an el't of"

<u>Ex's</u> 1 Let $E$ denote the set of even positive integers
- we also write
$$E = \{2, 4, 6, \ldots\}$$

Then $12 \in E$
$1 \notin E$
$-2 \notin E$.

2 - can denote finite sets by just writing all their el'ts in brackets
- called <u>roster notation</u>
- e.g. if $A = \{2, 4, 6, \pi\}$
  $B = \{\heartsuit, *, \pi\}$

  then $\pi \in A$ and $\pi \in B$
  while $\heartsuit \in B$ but $\heartsuit \notin A$.

↳ Sets are determined by their elements: order, repetition do not matter

e.g. if $A = \{1, 2, 3\}$
then $A = \{2, 1, 3\}$
$= \{1, 2, 3, 1\}$
also.

3) - sets can be elements of sets:
- if $A = \{1,2\}$ $B = \{3,4\}$
then $C = \{A, B\}$
$= \{\{1,2\}, \{3,4\}\}$ is
a legit set
- different from $D = \{1,2,3,4\}$
($C$ has 2 el'ts, $D$ has 4).

<u>Some fundamental sets</u>:

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ "natural numbers"
$\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$ "integers"
$\mathbb{Q} = \{m/n \mid m, n$ are in $\mathbb{Z}$ and $n \neq 0\}$ "rational numbers"

$\mathbb{R}$ = set of real numbers
$\mathbb{C}$ = set of complex numbers
$= \{a + bi \mid a, b$ are in $\mathbb{R}\}$.

So we have, e.g.,

$0 \notin \mathbb{N}$ but $0 \in \mathbb{Z}$
$3/4 \in \mathbb{Q}$ but $3/4 \notin \mathbb{Z}$
$\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$
$i \in \mathbb{C}$ but $i \notin \mathbb{R}$.

<u>Another</u> <u>important</u> <u>set</u>:

the <u>empty set</u> is the unique set with no elements.

- denoted $\{\}$ or $\emptyset$
- not the same as $\{\emptyset\}$
  - this set contains a single element, the empty set contains none.

## New sets from old ones

Set-builder notation: given a set $X$ and a well-defined property $P$, can form a set $Y$ consisting of all $x \in X$ with property $P$

We write:

always need to specify where $x$'s are being drawn from

$$Y = \{x \in X \mid x \text{ has } P\}$$

or $$Y = \{x \in X \mid P(x)\}$$

called "set-builder notation"

Ex's ① can define $E = \{2, 4, 6, \ldots\}$ by

$$E = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 2\}$$

or, more symbolically

$$E = \{n \in \mathbb{N} \mid \text{there is a } k \in \mathbb{N} \text{ s.t. } n = 2k\}$$

"such that"

② once $E$ is defined can use it to define other sets, e.g. let

$$O = \{n \in \mathbb{N} \mid \text{there is } k \in E \text{ s.t. } n = k-1\}$$
$$= \{1, 3, 5, \ldots\}$$

③ the set over which you range is important, e.g.

$$\{x \in \mathbb{R} \mid x^2 - 2 = 0\} = \{-\sqrt{2}, \sqrt{2}\}$$

but $\{x \in \mathbb{Z} \mid x^2 - 2 = 0\} = \emptyset$
since no integers satisfies $x^2 - 2 = 0$.

<u>Some more notation</u>:

- for a fixed $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$
- e.g. $[5] = \{1, 2, 3, 4, 5\}$.

<u>Subsets</u> - a set $Y$ is a <u>subset of</u> $X$ if for every $y \in Y$ we have $y \in X$

- we write $Y \subseteq X$.

- $Y$ is called a <u>proper subset</u> of $X$ if $Y \subseteq X$ but $Y \neq X$
- we write $Y \subsetneq X$ or $Y \subset X$

for "$Y$ is a proper subset of $X$"

**Ex's** ① $\{1,3\} \subseteq \{1,2,3,4\}$

why: $1 \in \{1,2,3,4\}$
and $3 \in \{1,2,3,4\}$

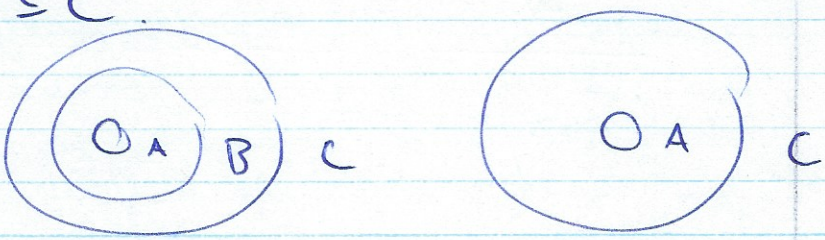
② $\{-1,3\} \not\subseteq \{1,2,3,4\}$

"is not a subset of"

why: $-1 \in \{-1,3\}$
$-1 \notin \{1,2,3,4\}$

③ $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

**Notice**: "$\subseteq$" is transitive, i.e. if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.




Let's prove this from the definition

**Prop'n 1** For any sets $A, B, C$, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Pf: – Suppose $x \in A$ is a fixed, arbitrary ele't of $A$
– Since $A \subseteq B$ we have $x \in B$
– Since $B \subseteq C$ we have $x \in C$
– Since $x \in A$ was arbitrary we have that every el't of $A$ is in $C$
– i.e. $A \subseteq C$.

using def'n of "subset"

## More ex's

(4) For any set $X$ we have $X \subseteq X$. (why?) (if $x \in X$ then $x \in X$ too...)

(5) Set-builder notation defines a subset, i.e. if $Y = \{x \in X \mid x \text{ has } P\}$ then $Y \subseteq X$

(6) For any set $X$ we have $\emptyset \subseteq X$
↳ automatic from def'n, but not obvious
↳ why: it is true that if (i) $x \in \emptyset$ then (ii) $x \in X$ simply because (i) never holds!

↳ more on this type of reasoning later.

## Equality of Sets

- a set is determined by its elements: two sets are the same exactly when they have the same elements

- can make this a precise def'n using $\subseteq$

<u>Def'n</u> For any sets $A, B$ we define $A = B$ if (and only if) $A \subseteq B$ <u>and</u> $B \subseteq A$

↓

i.e. $A = B$ iff ← "if and only if"
whenever $a \in A$ then $a \in B$ <u>and</u>
whenever $b \in B$ then $b \in A$

e.g. if $A = \{1, 2, 3\}$
$B = \{2, 1, 3\}$
then $A = B$

↳ main import of def'n is in <u>proofs</u>.

↳ to prove $A = B$ one shows:
(i) $A \subseteq B$ and
(ii) $B \subseteq A$.

↳ "double containment proofs"
see some of these soon.

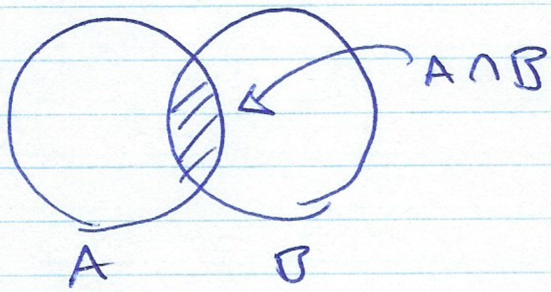
# Operations on Sets

## Intersections

<u>Def'n</u> Given sets $A, B$, the intersection of $A$ and $B$, denoted $A \cap B$, is the set of ele'ts belonging to <u>both</u> $A$ and $B$

i.e.

$x \in A \cap B$ iff $x \in A$ <u>and</u> $x \in B$.




<u>Ex</u> 1 if

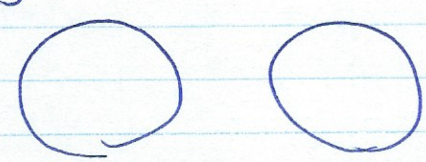
$$A = \{1, 2, 3, 4\}$$
$$B = \{1, 3, 5\}$$
$$C = \{2, 4, 6\}$$

then

$$A \cap B = \{1, 3\}$$
$$A \cap C = \{2, 4\}$$
$$B \cap C = \emptyset$$

<u>Def'n</u> Two sets are <u>disjoint</u> iff their intersection is $\emptyset$.

e.g. $B, C$ above are disjoint.



disjoint sets.

(2) <u>Prop'n</u> For any sets $A, B$ we have

(i) $A \cap B \subseteq A$
(ii) $A \cap B \subseteq B$

↳ "obvious" but let's practice proving from the def'ns

<u>Pf</u>: (i) - Sps $x \in A \cap B$ is arbitrary and fixed
- then by def'n of $\cap$ $x \in A$ <u>and</u> $x \in B$
- hence $x \in A$
- hence every $x \in A \cap B$ is an el't of $A$
- i.e. $A \cap B \subseteq A$

(ii) similar.

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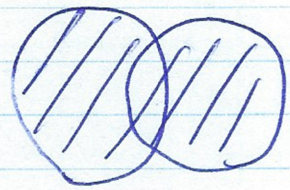
<u>Unions</u>

<u>Def'n</u> if $A, B$ are sets, the <u>union</u> of $A$ and $B$, denoted $A \cup B$, is the set of el'ts contained in either $A$ or $B$

i.e. $x \in A \cup B$ iff
$x \in A$ <u>or</u> $x \in B$.

- <u>Note</u>: "or" here (as in all math) is nonexclusive.

- i.e. $x \in A \cup B$ iff
  $x \in A$ or $x \in B$ or both



$A \cup B$

A B

<u>Ex's</u>

① ~~$\#$~~ $\{1,3,5\} \cup \{2,4,6\} =$
$\{1,2,3,4,5,6\} = [6]$

② if $E = \{2,4,6,\ldots\}$
$O = \{1,3,5,\ldots\}$
then $E \cup O = \mathbb{N}$

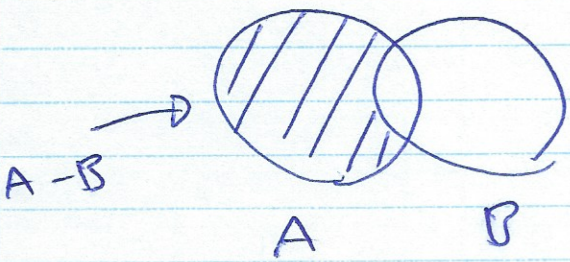
③ <u>Prop'n</u> For any sets $A, B$ we have
(i) $A \subseteq A \cup B$
(ii) $B \subseteq A \cup B$

<u>Pf</u>: try it yourself.

# Difference

**Def'n** if $A, B$ are sets, the difference of $A$ and $B$, denoted $A - B$, is the set of el'ts in $A$ that are not in $B$

i.e. $x \in A - B$ iff
$x \in A$ and $x \notin B$

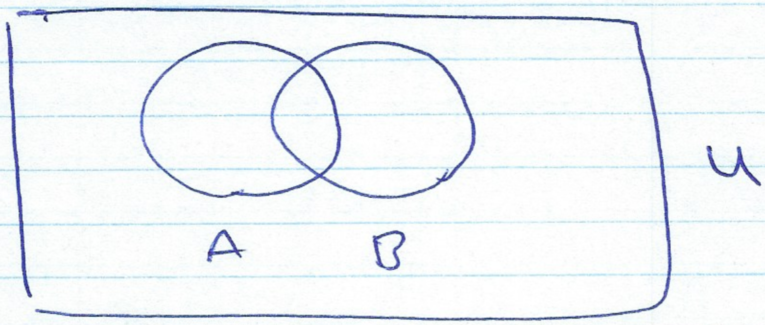



**Ex's** ① if $A = \{1,2,3\}$
$B = \{3,4,5\}$
then $A - B = \{1,2\}$
$B - A = \{4,5\}$

**Notice:** difference is not commutative, i.e. $A - B \neq B - A$ in general

However we always have $A \cup B = B \cup A$
and $A \cap B = B \cap A$

Note: in defining ∩ ∪ and −
it is sometimes convenient to assume that our sets A, B are both subsets of another set U (called a universal set)



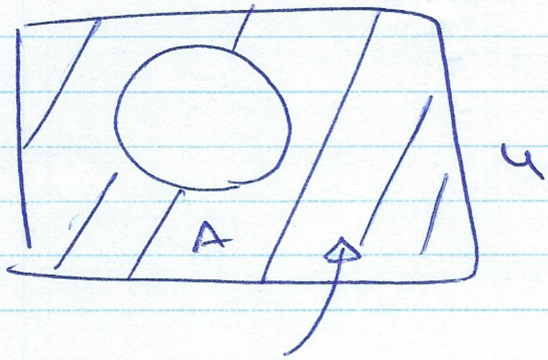

– then we can define these operations using set-builder notation:

$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
$$A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$$

## Complement

Def'n Given a set A and a universal set U s.t. $A \subseteq U$ the complement of A, denoted $\bar{A}$, is the set of el'ts in U that are not in A

$\bar{A} = \{x \in U \mid x \notin A\}$




$\bar{A}$

note: – $\bar{A}$ is only defined relative to $U$
– really $\bar{A} = U - A$

<u>Ex's</u> 1 Sps $U = \mathbb{N}$

$A = \{1,2,3\} = [3]$
$E = \{2,4,6,\ldots\}$
$O = \{1,3,5,\ldots\}$

<u>then</u>: $\bar{A} = \{4,5,6,\ldots\}$
$\bar{E} = \{1,3,5,\ldots\} = O$
$\bar{O} = \{2,4,6,\ldots\} = E$

# Indexing by Sets

- $\cap$ and $\cup$ allow us to combine <u>two</u> sets in certain ways.

- often useful to take unions/intersections of more than two sets

- need notation for indexing larger collections of sets.

<u>Ex</u>: - For any $i \in \mathbb{N}$, define

$$A_i = \{-i, 0, i\}.$$

so:

$$A_1 = \{-1, 0, 1\}$$
$$A_2 = \{-2, 0, 2\} \quad \text{etc.}$$

- Then $A_1 \cup A_2 = \{-2, -1, 0, 1, 2\}$

$$A_1 \cup A_2 \cup A_3 = \{-3, -2, -1, 0, 1, 2, 3\}$$

- or even

$$A_1 \cup A_2 \cup \cdots \cup A_{10} = \{-10, -9, \ldots, 8, 9, 10\}$$

- we might write above union more formally as

$$\bigcup_{i=1}^{10} A_i$$

- alternatively, instead of thinking of the index variable $i$ as "clicking up" from 1 to 10 can think of it as ranging over the set $[10] = \{1, 2, \ldots, 10\}$ and write the union as

$$\bigcup_{i \in [10]} A_i$$

This idea is useful:

Def'n Sps $I$ is a set (called an index set) s.t. for every $i \in I$ we have defined a set $A_i$

We define

$$\bigcup_{i \in I} A_i$$

as the set of el'ts contained in at least one of the $A_i$

i.e. $x \in \bigcup_{i \in I} A_i$

iff there is an $i \in I$ s.t. $x \in A_i$

We also define

$$\bigcap_{i \in I} A_i$$

as the set of el'ts contained in every $A_i$

i.e. $x \in \bigcap_{i \in I} A_i$ iff for every $i$ we have $x \in A_i$

[redacted]

<u>Ex's</u> For $i \in \mathbb{N}$ define $A_i = \{-i, 0, i\}$ as before.

① let $I = [10] = \{1, 2, \ldots, 10\}$

Then $\bigcup_{i \in I} A_i = \bigcup_{i \in \{1, \ldots, 10\}} A_i$

$= A_1 \cup A_2 \cup \cdots \cup A_{10}$

$= \{-10, -9, \ldots, -1, 0, 1, \ldots, 9, 10\}$

② An infinite union:

$\bigcup_{i \in \mathbb{N}} A_i = A_1 \cup A_2 \cup \cdots$

$= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$= \mathbb{Z}$.

③ let $E = \{2, 4, 6, \ldots\}$

Then $\bigcup_{i \in E} A_i = \{\ldots -4, -2, 0, 2, 4, \ldots\}$

④ OTOH :

on the other hand

$\bigcap_{i \in [10]} A_i = A_1 \cap A_2 \cap \cdots \cap A_{10}$

$= \{-1, 0, 1\} \cap \{-2, 0, 2\} \cap \cdots \cap \{-10, 0, 10\}$

$= \{0\}$

in fact already $A_1 \cap A_2 = \{0\}$

(5) Let $J = \{1,2,3\}$ and for every $j \in J$ define
$$B_j = \{j-2, j-1, j, j+1, j+2\}$$

so: $B_1 = \{-1, 0, 1, 2, 3\}$
$B_2 = \{0, 1, 2, 3, 4\}$
$B_3 = \{1, 2, 3, 4, 5\}$

hence: $\bigcup_{j \in J} B_j = \{-1, 0, 1, 2, 3, 4, 5\}$

whereas: $\bigcap_{j \in J} B_j = \{1, 2, 3\}$

(6) It may be that the indices themselves are sets!

e.g. Let
$$X = \{\{1,2\}, \{1,3\}, \{1,4\}\}$$

What is
$$\bigcup_{y \in X} y \ ?$$

The union of all sets in $X$:

i.e.
$$\bigcup_{y \in X} y = \{1,2\} \cup \{1,3\} \cup \{1,4\}$$
$$= \{1,2,3,4\}$$