Homework #9

1. Let $\mathcal{F}$ denote the set of *all* functions from $\mathbb{N}$ to $\mathbb{N}$, that is, $\mathcal{F} = \{f \subseteq \mathbb{N} \times \mathbb{N} \mid f \text{ is a function}\}$. Define a relation R on $\mathcal{F}$ by the rule $(f, g) \in R$ iff for every $n \in \mathbb{N}$ we have $f(n) \leq g(n)$. Prove that R is a partial order on $\mathcal{F}$.

2. Fix $m, n \in \mathbb{N}$. Define a mapping $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by $f([a]_n) = [a]_m$.

   a. Prove that if $m \mid n$ then f is a well-defined function. That is, prove that if $[a]_n = [b]_n$ then $f([a]_n) = f([b]_n)$.

   b. Let $n = 12$ and $m = 3$. Write $PreIm_f(\{[1]_3, [2]_3\})$ in roster notation.

   c. Suppose $m \nmid n$. Show that f is ill-defined. That is, show there exist $a, b \in \mathbb{Z}$ such that $[a]_n = [b]_n$ but $f([a]_n) \neq f([b]_n)$.

3. Suppose that A, B, and C are nonempty sets and $f : A \to B$ and $g : B \to C$ are functions.

   a. Prove that if f and g are surjections then so is $g \circ f$.

   b. Prove that if f and g are injections then so is $g \circ f$.

   c. Use your results from parts (a.) and (b.) to prove that if f and g are bijections then so is $g \circ f$.

4. Suppose X and Y are nonempty sets and $f : X \to Y$ is a function. Define a new function $F : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $F(B) = PreIm_f(B)$. Prove that F is injective if and only if f is surjective.