Homework #6

1. Let $P(n)$ be a variable proposition. In each of the following cases, assume that both BC (the base case(s)) and IH (the inductive hypothesis) hold. Determine the largest subset $S \subseteq \mathbb{Z}$ for which, from these assumptions, we can conclude $(\forall n \in S)P(n)$.

   a. BC: $P(-3)$. IH: $(\forall n \in \mathbb{Z})(P(n) \Rightarrow P(n+1))$.
   b. BC: $P(1)$. IH: $(\forall n \in \mathbb{N})(P(n) \Rightarrow P(2n))$.
   c. BC: $P(0)$. IH: $(\forall n \in \mathbb{Z})(P(n) \Rightarrow P(n-1) \wedge P(n+1))$.
   d. BC: $P(0) \wedge P(1)$. IH: $(\forall n \in \mathbb{Z})(P(n) \Rightarrow P(n+3))$.

2. For $n \in \mathbb{N} \cup \{0\}$, let f_n denote the nth Fibonacci number. Prove that

$$\sum_{k=0}^{n} f_{2k} = f_{2n+1} - 1.$$

3. In class we proved that if n is a multiple of 3, then f_n is even. Prove the converse of this statement. That is, prove that if n is not a multiple of 3, then f_n is odd.

4. Prove by induction that the number of subsets of $[n]$ is exactly 2^n.

5. Define a sequence a_n recursively, as follows:

$$a_0 = 4, a_1 = 9, \text{ and } a_n = 5a_{n-1} - 6a_{n-2} \text{ for all } n \geq 2.$$

   Use strong induction to prove that, for all $n \in \mathbb{N} \cup \{0\}$, we have $a_n = 3 \cdot 2^n + 3^n$.

***

XC. (1 point) In this problem, we will prove by induction that all cats are black. Let $P(n)$ be the proposition, "In any group of n cats, if at least one of the cats in the group is black, then all of the cats in the group are black."

Claim: $(\forall n \in \mathbb{N})P(n)$ holds.

Proof: (BC) $P(1)$ is clearly true.

(IH) Fix $n \in \mathbb{N}$ and assume $P(n)$ is true.

(IS) We prove $P(n+1)$. Suppose we have a group of $n+1$ cats that includes at least one black cat. Call this cat Midnight. Now, fix a different cat in the group. Call this cat Snow. Remove Snow from the group. We are left with a group of n cats that includes a black cat (namely, Midnight). By our inductive hypothesis, all cats in this group are black.

Now remove Midnight and return Snow to the group. Again we have a group of n cats. All the cats in this group are black, except perhaps for Snow. Applying our inductive hypothesis to this group, we see that in fact all cats in this group must be black, including Snow. Hence all $n+1$ of the cats in our original group are black.

We have proved that if a group of $n+1$ cats contains a black cat, then all cats in the group are black. That is, we have proved $P(n+1)$ holds.

By induction we have proved $(\forall n \in \mathbb{N})P(n)$ holds.

Now consider the group of *all* cats. (This group is of size N, where N is approximately 600 million.) Since there is at least one black cat, all cats must be black, by our claim.

Why is this argument flawed? Give a one or two sentence answer.