Homework #2

1. Let $(a, b) \in \mathbb{R}^2$ and fix $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. Define $C_{(a,b),\epsilon}$ as the set of real numbers "within ϵ" of (a, b):

$$C_{(a,b),\epsilon} = \{(x, y) \in \mathbb{R}^2 | \sqrt{(x-a)^2 + (y-b)^2} < \epsilon\}.$$

   a.) Give a geometric description of $C_{(a,b),\epsilon}$.

   b.) Identify the following sets. Write your answer in the form of $C_{(a,b),\epsilon}$ or as one of the standard sets discussed in class.

      i. $C_{(0,0),1} \cap C_{(0,0),2}$

      ii. $C_{(0,0),1} \cup C_{(0,0),2}$

      iii. $C_{(0,0),1} \cap C_{(2,2),1}$

   c.) For a given $\epsilon > 0$, define $D_{(a,b),\epsilon}$ as follows:

$$D_{(a,b),\epsilon} = \{(x, y) \in \mathbb{R}^2 | \sqrt{(x-a)^2 + (y-b)^2} \leq \epsilon\}.$$

   What is $D_{(a,b),\epsilon} - C_{(a,b),\epsilon}$ geometrically? Write a definition for this set using set-builder notation.

2. Let A, B, and C be sets. Prove that

$$A - (B - C) \subseteq (A - B) \cup C$$

   and then provide an example of sets A, B, and C for which the containment is *strict*.

3. Let A and B be sets, and suppose that $\mathcal{P}(A) = \mathcal{P}(B)$. Is it necessarily the case that $A = B$? If so, prove it. If not, provide a counterexample.

4. For each $n \in \mathbb{N}$, let $A_n = [n] \times [n]$. Define $B = \bigcup_{n \in \mathbb{N}} A_n$. Does $B = \mathbb{N} \times \mathbb{N}$? Either prove that it does, or show why it does not.

5. Let $I = \{x \in \mathbb{R} \,|\, 0 < x < 1\}$. For each $x \in I$, define $S_x = \{y \in \mathbb{R} \,|\, x < y < x + 1\}$. Provide a double containment proof that

$$\bigcup_{x \in I} S_x = \{z \in \mathbb{R} \,|\, 0 < z < 2\}.$$