9.2 Direction Fields

- we've seen: how to solve diff 'eq'n
  of form $y' = f(x)g(y)$
- "separable eq'ns"
- e.g. we showed that:
  $y' = 2y$
  has family of solns:
  $y = Ae^{2x}$
- let's graph $y$ vs $x$ for $A = -3, -2, -1, 1, 2$
If we draw some mini tan line at various points along curves and then erase curves, retain a "slope field" for our Family.

Observe: can determine slopes of these tan line from original eq'n.
So can use this idea to sketch solns. to diff' eq's we can't solve explicitly.

Ex: consider eq'n
\[ y' = x + y \]
- not a separable eq'n; instead of solving for the family of sol'n we sketch their graphs using a slope field.

- the point: if \( y = f(x) \) is sol'n passing thru \((x_0, y_0)\), eq'n tells us tan line to \( f \) there has slope \( x_0 + y_0 \).

- so, e.g. sol'n through \((0,1)\) has slope 
  \[
  \begin{align*}
    x + y &= 1 \\
    0 + 1 &= 1 \\
    0 + 2 &= 2 \\
    (1) &= (1) \\
    (1) &= (1)
  \end{align*}
  \]

Table of slopes:

\[
\begin{array}{c|cccccc}
 x  & -3 & -2 & -1 & 0 & 1 & 2 \\
\hline
 y  & 1 & -2 & -1 & 0 & 1 & 2 \\
 z  & 2 & -1 & 0 & 1 & 2 & 3 \\
 u  & 3 & 0 & 1 & 2 & 3 & 4 \\
 w  & 4 & 1 & 2 & 3 & 4 & 5 \\
 v  & 5 & 1 & 2 & 3 & 4 & 5 \\
 w  & 6 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
Slope field:

Graph of $y = f(x)$ thru
(0,0) satisfying
$y' = x + y$ has
slope $0 \leftrightarrow 0$
there

- Can get rough sketch for a
  particular sol'n $y = f(x)$ going thru
  (e.g.) (0,0) by "following field lines."

Note: actually, family of solns given
by: $y = ce^{x} - x - 1$
ex: a) sketch slope field for eq'n $y' = \cos^2(y)$ for $-\pi/2 \leq y \leq \pi/2$
$-2 \leq x \leq 2$

b) solve eq'n explicitly for sol'n thru $(0,0)$.

Sol'n: a)

\[ \begin{array}{c|cccccc}
    & -2 & -1 & 0 & 1 & 2 \\
\hline
-\pi/2 & 0 & 0 & 0 & 0 & 0 \\
-\pi/4 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0       & 0 & 1 & 1 & 1 & 1 \\
\pi/4   & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\pi/2   & 0 & 0 & 0 & 0 & 0 \\
\end{array} \]

Notice: $y'$ depends only on $y$.
b) eqn is separable:
\[
\frac{dy}{dx} = \cos^2 y
\]
\Rightarrow \frac{1}{\cos^2 y} \ dy = \ dx
\Rightarrow \sec^2 y \ dy = \ dx
\Rightarrow \int \sec^2 y \ dy = \int \ dx
\Rightarrow \tan(y) = x + C

If \( x = y = 0 \) then \( \tan(0) = 0 + C \)
\Rightarrow \ C = 0

So soln thru \( (0,0) \) is given by \( \tan(y) = x \)
\Rightarrow \ y = \tan^{-1}(x) \ \checkmark