\[ \int \sqrt{3-2x-x^2} \, dx \\
= \int \sqrt{-\left(x^2+2x-3\right)} \, dx \\
= \int \sqrt{-\left[(x+1)^2 - 4\right]} \, dx \\
= \int \sqrt{4 - (x+1)^2} \, dx \\
\]

Let \( x + 1 = 2 \sin \theta \)
\[ dx = 2 \cos \theta \, d\theta \]
\[ = \int \sqrt{(2 \cos \theta)^2} \, d\theta \]
\[ = \int 4 \cos^2 \theta \, d\theta \]
\[ = 2 \int 1 + \cos 2\theta \, d\theta \]
\[ = 2 \left( \theta + \frac{1}{2} \sin 2\theta \right) \]
\[ = 2 \left( \theta + \sin \theta \cos \theta \right) \]
\[ = 2 \left( \sin^{-1} \left( \frac{x+1}{2} \right) + \frac{(x+1)\sqrt{4-(x+1)^2}}{4} \right) + C \]
\[ \int \sqrt{1 - \sin x} \, dx \]

\[ = \int \sqrt{\frac{(1-\sin x)(1+\sin x)}{1+\sin x}} \, dx \]

\[ = \int \sqrt{\frac{1-\sin^2 x}{1+\sin x}} \, dx \]

\[ = \int \sqrt{\frac{\cos^2 x}{1+\sin x}} \, dx \]

\[ = \int \frac{\cos x}{\sqrt{1+\sin x}} \, dx \]

Let \( u = 1 + \sin x \)

\( du = \cos x \, dx \)

\[ = \int \frac{1}{\sqrt{u}} \, du \]

\[ = 2\sqrt{u} + C \]

\[ = 2\sqrt{1+\sin x} + C \]
7.7 Approximating Integrals

- Many integrable functions for which \( \int_a^b f(x) \, dx \) can't be determined (in terms of elementary functions).
- Can still get good estimates of \( \int_a^b f(x) \, dx \) using Riemann sums.
- Come in various flavors:
  - Left endpoint approximation
  - Right endpoint approximation
  - Midpoint rule
  - Trapezoidal rule
Left e.p.

\[ y = f(x) \]

Right e.p.

\[ R_n = \sum_{i=1}^{n} f(x_i) \Delta x \]

\[ = \Delta x (f(x_1) + f(x_2) + \ldots + f(x_n)) \]

Midpoint

\[ M_n = \sum_{i=1}^{n} f(\frac{x_i + x_{i-1}}{2}) \Delta x \]

\[ = \frac{\Delta x}{2} \left( f(x_1) + f(x_2) + \ldots + f(x_n) \right) \]

Trapezoid

\[ T_n = \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i+1})}{2} \Delta x \]

\[ = \frac{\Delta x}{2} \left( f(x_1) + 2f(x_2) + \ldots + f(x_n) \right) \]
all of these methods yield the integral in the limit

$$\lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n = \lim_{n \to \infty} M_n = \lim_{n \to \infty} T_n$$

$$= \int_a^b f(x) \, dx$$

- but, for a given $n$, some better approximate $\int_a^b f(x) \, dx$ than others.

ex: Use trapezoidal rule and midpoint rule to approximate area under $y = \frac{1}{x}$ between $x=1$ and $2$. Use $n=5$.

$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$

$x_0 = 1, 1.2, 1.4, 1.6, 1.8, 2 = x_5$

$x_1, x_2, x_3, x_4$
midpoints: $1.1, 1.3, 1.5, 1.7, 1.9$

Trap rule gives:

$$
\int_1^2 \frac{1}{x} \, dx \approx T_5 = \frac{\Delta x}{2} (f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2))
$$

$$
= \frac{2}{2} \left( \frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2} \right)
$$

$$
= .6956 \ldots
$$

Midpoint gives:

$$
\int_1^2 \frac{1}{x} \, dx \approx M_5 = \Delta x \left( f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right)
$$

$$
= .2 \left( \frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)
$$

$$
= .6919 \ldots
$$

If we actually compute:

$$
\int_1^2 \frac{1}{x} \, dx = \left( \ln |x| \right)^2 \bigg|_1^2 = \ln(2) - \ln(1) = \ln(2)
$$

$$
= .6931 \ldots
$$
trap rule overshoots, m.p. undershoots
m.p. a bit closer overall

In general: trap and m.p. tend to be better than r.e.p. and i.e.p.

m.p. tends to be better than trap.

to make these statements precise, we define the errors:

\[ E_T = \int_c^b f(x) \, dx - T_n \]
\[ E_M = \int_c^b f(x) \, dx - M_n \]
\[ E_L = - \ln \]
\[ E_R = - R_n \]

\[ E_M \text{ tends to be smaller (in absolute value) than } E_T, \text{ in following sense:} \]

\[ \text{e.g. for above:} \]
\[ E_T = \int_1^2 \frac{1}{x} \, dx - T_5 \]
\[ = \ln 2 - 0.6936 \approx -0.6936 \]
\[ E_M = \int_1^2 \frac{1}{x} \, dx - M_5 \]
\[ = 0.012... \]
**Theorem**: Suppose \(|f''(x)| \leq k\) on the interval \(a \leq x \leq b\). Then:

\[ |E_T| \leq \frac{k(b-a)^3}{12n^2} \quad \text{and} \quad |E_m| \leq \frac{k(b-a)^3}{24n^2} \]

Note: these are upper bound estimates for the errors — actual errors may be smaller.

**Ex:** Consider our estimates \(M_5\) and \(T_5\) for \(\int_1^2 \frac{1}{x} \, dx\)

Observe:

\[ \frac{d^2}{dx^2} \frac{1}{x} = \frac{d}{dx} \frac{-1}{x^2} = \frac{2}{x^3} \]

\[ \left| \frac{2}{x^3} \right| \text{ is maximized at } x = 1 \text{ on } (1, 2) \]

\[ \Rightarrow \left| \frac{2}{x^3} \right| \leq \left| \frac{2}{1^3} \right| = 2 \text{ on } (1, 2) \]

So theorem says:

\[ |E_T| \leq \frac{2(2-1)^3}{12.5^2} \quad |E_m| \leq \frac{2(2-1)^3}{24.5^2} \]

\[ n = 0.0066 \ldots \Rightarrow 0.0037 \]

but actual errors were:

\[ E_T = -0.0025 \ldots \quad E_m = 0.0012 \ldots \]
Ex: How large must $n$ be to guarantee (using theorem) that $|E_T|$ and $|E_m|$ are $\leq .0001$ for $\int_1^x \frac{1}{x} \, dx$.

So/\n: By theorem:

$$|E_T| \leq \frac{2(1)^3}{12n^2}$$

want:

$$\leq .0001$$

$$\Rightarrow \frac{1}{6} \leq .0001n^2 \Rightarrow \frac{\frac{1}{6}}{.0001} \leq n^2$$

$$\Rightarrow n > 40.8$$

So $n$ must be at least 41 to guarantee desired error for trap rule.

For $|E_m|$ we know:

$$|E_m| \leq \frac{2(1)^7}{24n^2}$$

want:

$$\leq .0001$$

$$\Rightarrow n^2 \geq \frac{1/12}{.0001}$$

$$\Rightarrow n \geq 28.86$$

So $n \geq 29$ guarantees desired error.