10.3 Polar Coordinates

- A point P in the plane is uniquely determined by its rectangular coordinates (x, y).

\[ y \rightarrow P = (x, y) \]
\[ \downarrow \]
\[ x \]

- Can also specify P by its polar coordinates \((r, \theta)\) where:
  - \(r\) = distance to origin
  - \(\theta\) = angle made w/ x-axis

\[ r \quad \theta \]
\[ \uparrow \]
\[ \downarrow \]

**Example:**

\( P = (2, \pi/4) \)
\( Q = (2, 5\pi/4) \) are shown below.
Note: we allow $\theta > 2\pi$ and $\theta < 0$

e.g.: $P$ also has coords $(2, \frac{\pi}{4})$ and $(2, \frac{7\pi}{4})$

So: polar coords are not unique.

Note: also allow $r < 0$.

e.g.: $Q$ also has coords $(-2, \frac{\pi}{4})$
Translating:

from polar to rect: use \( x = r \cos \theta \)
\( y = r \sin \theta \)

from rect. to polar: use \( x^2 + y^2 = r^2 \)
\( \tan \theta = \frac{y}{x} \)

ex: if \( P \) has polar coords \((2, \frac{\pi}{4})\)
then \( P \) has rectangular coords
\[
\begin{align*}
x &= 2 \cos \left( \frac{\pi}{4} \right) = 2 \left( \frac{\sqrt{2}}{2} \right) = \sqrt{2} \\
y &= 2 \sin \left( \frac{\pi}{4} \right) = 2 \left( \frac{\sqrt{2}}{2} \right) = \sqrt{2}
\end{align*}
\]
If P has rectangular coords (3,4) then P has polar coords given by:

\[ r^2 = x^2 + y^2 = 25 \Rightarrow r = 5 \]
\[ \theta = \tan^{-1} \left( \frac{y}{x} \right) = 0.927 \ldots \]

(5, 0.927\ldots)

**Polar curves**

- can also specify curves w/ polar eqns.
- usually we consider equations of the form \( r = f(\theta) \), i.e. specifying \( r \) as a function of \( \theta \).
- graph of such an eq'n is all polar \((r, \theta)\) where \( r = f(\theta) \)

ex: graph the polar curve \( r = 2 \).

sol'n: consists of all points \((r, \theta)\) where \( r = 2 \).
to graph more complicated curves

$r = f(\theta)$, various approaches:

- plot points (not usually effective by itself)
- translate to rectangular coords (doesn't always work)
- use calculus to find points w/ none or vert. tan lines (see second ex below)
- use brain

**Ex:** graph $r = 2 \cos \theta$

**Sol'n:** can plot some points to start.
\[ r = 2 \cos \theta \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( \frac{\pi}{4} )</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{3\pi}{4} )</td>
<td>(-\sqrt{2} )</td>
</tr>
<tr>
<td>( \pi )</td>
<td>(-2 )</td>
</tr>
</tbody>
</table>

\[ (\frac{\sqrt{2}}{2}, \frac{\pi}{4}) \]
\[ (-2, \pi) \]

Only gives a rough sense of curve.

In this case we can translate to rectangular coordinates, but requires some creativity.

Use:
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ x^2 + y^2 = r^2 \]

to get \( r = 2 \cos \theta \) into \( x, y \)

\[ \Rightarrow \cos \theta = \frac{x}{r} \]
\[ \Rightarrow 2 \cos \theta = 2 \frac{x}{r} = r \]
\[ \Rightarrow r^2 = 2x \]
\[ \Rightarrow x^2 + y^2 = 2x \]
\[ \Rightarrow x^2 - 2x + y^2 = 0 \]

(complete square)
\[ \Rightarrow x^2 - 2x + 1 + y^2 = 1 \]
\[ \Rightarrow (x-1)^2 + y^2 = 1 \]

Circle of radius 1 centered at \((1,0)\).
Using calculus: to graph a curve $r = f(\theta)$, finding $\frac{\text{dy}}{\text{dx}}$ at various points gives more reliable info than plotting points randomly and trying to interpolate.

So, we need a formula for $\frac{\text{dy}}{\text{dx}}$.

Using $x = r \cos \theta = f(\theta) \cos (\theta)$,

$y = r \sin \theta = f(\theta) \sin (\theta)$

(idea: can view a polar curve $r = f(\theta)$ as a parametric curve (using $\theta$ as a parameter instead of $t$))

- We have: $\frac{\text{dx}}{\text{d}\theta} = \frac{\text{dr}}{\text{d}\theta} \cos \theta - r \sin \theta$

- $\frac{\text{dy}}{\text{d}\theta} = \frac{\text{dr}}{\text{d}\theta} \sin \theta + r \cos \theta$

- From our previous formula we know:

$\frac{\text{dy}}{\text{dx}} = \frac{\text{dy/d}\theta}{\text{dx/d}\theta} = \frac{\frac{\text{dr}}{\text{d}\theta} \sin \theta + r \cos \theta}{\frac{\text{dr}}{\text{d}\theta} \cos \theta - r \sin \theta}$