Taylor and Maclaurin Series

The big Q: Which functions \( f \) have power series rep'ns? How do we find them?

So far: started w/ \( f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) for \(-1 < x < 1\)

and found power series rep'ns for various ones on the function: \( \frac{1}{1+x^2}, \ln(1+x), \text{etc.} \)

A new approach: Sups we are given a function \( f(x) \) and we assume \( f(x) \) has a power series rep'n:

\[ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots \]

on some interval of the form \((a-R, a+R)\)

But we don't know the \( c_n \)'s:

How do we find them?

Observe: by our differentiation rules for power series, we have:
\[ f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \]

so: \[ f'(a) = c_1 = 1 \cdot c_1 \]

\[ f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \ldots \]

so: \[ f''(a) = 2c_2 = 2 \cdot c_2 \]

\[ f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + 5 \cdot 4 \cdot 3c_5(x-a)^2 + \ldots \]

so: \[ f'''(a) = 3 \cdot 2c_3 = 3 \cdot c_3 \]

and in general we see:

\[ f^{(n)}(x) = n! \cdot c_n + \text{terms with} \ (x-a) \]

so: \[ f^{(n)}(a) = n! \cdot c_n \]

we get a formula for \( c_n \):

\[ c_n = \frac{f^{(n)}(a)}{n!} \]

Thus is big news: if we can find derivatives \( f^{(n)}(a) \), we can solve for the coefficients \( c_n \) in \( f \)'s power series rep's \( a \) (assuming such a rep's exists)
We've proved:

**Theorem:** If \( f(x) \) has a power series representation centered at \( a \), i.e., if there is \( R > 0 \) such that

\[
f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a| < R
\]

then the coefficients \( c_n \) are given by:

\[
c_n = \frac{f^{(n)}(a)}{n!}
\]

So then:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \text{for } |x-a| < R
\]

\[
= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \ldots
\]

- this series is called the **Taylor series** for \( f(x) \) centered at \( a \).
- In the case when \( a = 0 \), called the **Maclaurin series** for \( f \).
Thm says: IF \( f(x) \) can be represented by a power series @ \( x=a \), then the power series is given by the Taylor series.

- Some f's do not have a power series rep'n anywhere; such f's will not equal their Taylor series anywhere.
- We will typically ignore the question: "does this f(x) have a Taylor series expansion @ \( x=a \)?"
  and just assume that it does.

- But you should be aware this assumption is being made.
  (see book for a way to prove a given f has a Taylor series)

Ex: Find the Maclaurin series for \( f(x) = e^x \) and its radius of convergence.

Sol'n: By thm, Maclaurin series given by:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n
\]
In this case \( f(x) = e^x \).

So \( f'(x) = f''(x) = f'''(x) = \ldots = e^x \).

Hence \( f^{(n)}(0) = e^0 = 1 \) for every \( n \).

So Maclaurin series is:

\[
\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]

we've checked before: this series converge everywhere (i.e. radius is \( \infty \)).

Can be proved: \( e^x \) equals its

Maclaurin series, i.e.

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots
\]

for every \( x \).

In particular:

\[
e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots
\]

Can use this expression to get approximations for \( e \).
For example: \[ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708\ldots \]

whereas \( e = 2.718\ldots \)

A given function \( f(x) \) can have different power series rep's around different centers \( a \).

**Ex:** Find the Taylor series for \( e^x \) around \( a = 2 \).

**Sol:** by theorem, Taylor series given by:

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n
\]

Here \( f(x) = e^x \) so \( f^{(n)}(x) = e^x \) so \( f^{(n)}(2) = e^2 \) for every \( n \).

Taylor series is:

\[
\sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n = e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2
\]

\[+ \frac{e^2}{3!} (x-2)^3 + \ldots \]

\( e^2 \) to check: series converges everywhere and can prove \( e^x = e^x \) everywhere.

Hence \( e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n \) for every \( x \)