Ex: Consider the curve $C$
defined by $x = t^2$
$y = t^3 - 3t$

(a) Show that $C$ has two tangents
at the point $(3, 0)$ and find their slopes.

Solve: observe: $y = 0 \Rightarrow t^3 - 3t = 0$
$\Rightarrow t(t^2 - 3) = 0$
$\Rightarrow t = 0$ or $t = \sqrt{3}$
or $t = -\sqrt{3}$

$\Rightarrow$ curve crosses $x$-axis at these $t$'s.

e.g. $t = 0, \quad x = 0^2 = 0$
$t = \sqrt{3}, \quad x = (\sqrt{3})^2 = 3$
$t = -\sqrt{3}, \quad x = (-\sqrt{3})^2 = 3$

So: curve has $(3, 0)$ at $t = \pm \sqrt{3}$.
It crosses itself at this point.

we have: $\frac{dy}{dt} = y'(t) = 3t^2 - 3$
$\frac{dx}{dt} = x'(t) = 2t$

So: $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$
$\Rightarrow t = \sqrt{3} \rightarrow \frac{\sqrt{3} \cdot \sqrt{3}}{2 \sqrt{3}} = \frac{3}{2}$
$\Rightarrow t = -\sqrt{3} \rightarrow \frac{-3}{-2 \sqrt{3}} = \frac{\sqrt{3}}{2}$
(b) Find the points on the curve with horizontal or vertical tangent lines.

\[
\frac{dy}{dx} = 0 \quad \frac{dy}{dx} = \pm \infty
\]

**Solution:** horizontal: we solve \( \frac{dy}{dx} = 0 \)

\[ \Rightarrow \frac{3t^2 - 3}{2t} = 0 \]

\[ \Rightarrow 3t^2 - 3 = 0 \Rightarrow t^2 - 1 = 0 \]

\[ \Rightarrow (t - 1)(t + 1) = 0 \]

\[ \Rightarrow t = \pm 1 \]

@ \( t = 1 \): \( x = t^2 = 1, \ y = 1^3 - 3(1) = -2 \)

@ \( t = -1 \): \( x = (-1)^2 = 1, \ y = (-1)^3 - 3(-1) = 2 \)

So we have horizontal tangents at \((1, -2)\)

and \((1, 2)\) (marked)

**Vertical:** solve: \( \frac{dy}{dx} = \pm \infty \) i.e.

\[ \Rightarrow \frac{3t^2 - 3}{2t} = \pm \infty \Rightarrow 2t = 0 \Rightarrow t = 0 \]

really I mean:

\[ \lim_{t \to 0} \frac{3t^2 - 3}{2t} = -\infty \]

@ \( t = 0 \): \( x = 0^2 = 0, \ y = 0^3 - 3(0) = 0 \)

So: vertical tangent at \((0, 0)\)
Second derivative: we can again reason using chain rule to get a formula for $\frac{d^2y}{dx^2}$ in terms of t's:

Chain rule says:

$$\frac{d}{dt} \left( \ldots \right) = \frac{d}{dx} \left( \ldots \right) \frac{dx}{dt}$$

So:

$$\frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dx} \right) \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dy^2}{dx^2} = \frac{dt}{dx} \left( \frac{dy}{dx} \right) \cdot \frac{dx}{dt}$$

ta da!

(c) Find where the curve C above is concave up and concave down.

$$\frac{d^2y}{dx^2} \geq 0 \quad \frac{d^2y}{dx^2} \leq 0$$

Solution: First we find $\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right)$

$$= \frac{d}{dt} \left( \frac{3t^2-3}{2t} \right)$$

$$= \frac{dt}{dt} \left( \frac{3t^2-3}{2t} \right)$$
\[
= \frac{2t(6t) - (3t^2 - 3)2}{(2t)^2} \\
= \frac{2t}{12t^2 - 6t^2 + 6} \\
= \frac{6t^2 + 6}{8t^3} = \frac{3t^2 + 3}{4t^3}
\]

Since numerator always positive, second deriv is:
- \( \geq 0 \) when \( 4t^3 > 0 \)
- \( < 0 \) when \( 4t^3 < 0 \)
- \( \geq 0 \) when \( t > 0 \)
- \( < 0 \) when \( t < 0 \)

(c) we can now sketch curve:

- Concave up \( t > 0 \)
- Concave down \( t < 0 \)
Arc Length:

**Theorem** If a curve C is parameterized by:
\[
\begin{align*}
    x &= f(t) \\
    y &= g(t)
\end{align*}
\]
and this parameterization does not overlap itself (except perhaps at isolated points) for \( \alpha \leq t \leq \beta \) then the length of C over \( \alpha \leq t \leq \beta \) is given by:
\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

A formula can be derived in a similar way to our previous arc length formula:
\[
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \quad \text{for} \quad y = f(x) \quad \text{over} \quad a \leq x \leq b
\]

See book for details.

**Ex:** Find the circumference of the unit circle using the parameterization
\[
\begin{align*}
    x &= \cos (\pi t) \\
    y &= \sin (\pi t)
\end{align*}
\]
Solve: this parametrization traverses the circle exactly once as \( t \) goes from 0 to 2:

\[
\begin{align*}
t &= 0, 1, ½, 3½ \\
\end{align*}
\]

we have

\[
\begin{align*}
\frac{dx}{dt} &= -\pi \sin (\pi t) \\
\frac{dy}{dt} &= \pi \cos (\pi t) \\
\end{align*}
\]

so by our formula:

\[
\begin{align*}
L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\
&= \int_0^2 \sqrt{\pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t)} \, dt \\
&= \int_0^2 \pi \sqrt{\sin^2(\pi t) + \cos^2(\pi t)} \, dt \\
&= \int_0^2 \pi \, dt \\
&= \pi \left[ t \right]_0^2 = 2\pi \\
\end{align*}
\]
Notice: if we instead integrated from $t=0$ to $t=4$ we get:

$$\int_0^4 11 \, dt$$

$$= 44\pi$$

Thus corresponds to the length of the path traversing the circle twice.