If: \( \lim_{n \to \infty} b_n = 0 \) and \( b_{n+1} \leq b_n \) for every \( n \),
then: \( \sum (-1)^n b_n \) converges.

E.g. \( \sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \)
converges since \( \lim_{n \to \infty} \frac{1}{n} = 0 \) and \( \frac{1}{n+1} < \frac{1}{n} \).

Also useful: Alt. Dursu estimation: if \( \sum (-1)^n b_n \) satisfies 0 and 2, then:
\( S_n = b_1 - b_2 + b_3 - \cdots \pm b_{n+1} \)
\( \) is with \( |b_{n+1}| \) of entire series \( \sum (-1)^n b_n \)
(i.e. \( |S_{n+1}| \leq b_{n+1} \)).

E.g. \( 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.83 \)
is win \( \frac{1}{4} \) of \( \sum \frac{(-1)^{n+1}}{n} \) (in fact \( \sum \frac{(-1)^{n+1}}{n} = \ln(2) = 0.693 \ldots \)).

(vi) ratio and root tests:
Def'n: \( \sum a_n \) is absolutely convergent
if \( \sum |a_n| \) converges.
Fact: absolute convergence implies convergence (but not other way around).

Ex: Sup \( \sum a_n \) is a series, and consider \( \sum |a_n| \). Let \( S_n = |a_0| + |a_1| + \ldots + |a_n| \) be the \( n \)th partial sum. Assume \( S_n \leq 1 \) for all \( n \).

Does \( \sum |a_n| \) converge?

A: yes! the sequence \( S_n \) is:

- increasing, since \( S_{n+1} = S_n + |a_{n+1}| \geq S_n \)
- bounded (by 1, by assumption)

Hence \( S_n \) converges by monotone convergence.

i.e. \( \lim_{n \to \infty} S_n = L \) exists.

Hence by def'n, \( \sum |a_n| \) converges.

Hence by fact, \( \sum a_n \) converges.

Note: ratio/root tests give absolute convergence.

E.g. \( \sum \frac{(-1)^{n-1}}{n!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \)

converges, since \( \lim_{n \to \infty} \left| \frac{(-1)^n \frac{1}{(n+1)!}}{(-1)^{n-1} \frac{1}{n!}} \right| = \lim_{n \to \infty} \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{1}{n+1} = 0 < 1 \).
as does \( \sum \frac{1}{n^n} \) since
\[
\lim_{n \to \infty} \frac{1}{n^n} = \lim_{n \to \infty} \frac{1}{n^n} = 0 < 1.
\]

**Power Series**: a power series (centered @ \( x = a \)) is written:
\[
\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots
\]

E.g. \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n = 1 - \frac{1}{2} x + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \ldots \)

is a power series.

A power series is not a series. Become a series if we specify \( x \).

E.g. if \( x = 2 \) in above we get:
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} 2^n = 1 - \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 2^2 - \frac{1}{4} \cdot 2^3 + \ldots
\]

(diverges)

**The question**: for which \( x \)'s does \( \sum c_n (x-a)^n \) converge? Use ratio/root test to answer.

E.g. For series above:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}/n+2}{(-1)^n/n+1} \right| x^{n+1}
\]
\[
= \lim_{n \to \infty} \frac{n+1}{n+2} |x| = |x|
\]
By ratio test: series converges if \(|x|<1\) and diverges if \(|x|>1\) and if \(x=1\), I have to check by hand (you try)

On its interval of convergence, a power series defines a function:
\[
\sum_{n=0}^{\infty} c_n(x-a)^n = f(x) \quad \text{on} \quad (a-R, a+R)
\]

Can think of \(f(x)\) as an "infinite polynomial." Differentiation and integration behave as we expect.

If \(f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots\)

then \(f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}\)

and \(\int f(x) \, dx = c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \ldots + c = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + c\)

Amazing fact: many well-known functions have power series reprs (at least in parts of their domain)

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad x \in (-1, 1)
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x
\]

\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \quad \text{all } x
\]
We can use these reps'n to get:

\[
\frac{1}{(1-x)^2} = \frac{d}{dx} \left( \frac{1}{1-x} \right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n \quad x \in (-1,1)
\]

\[
= \frac{d}{dx} (1 + x + x^2 + \ldots) = 1 + 2x + 3x^2 + \ldots
\]

\[
= \sum_{n=1}^{\infty} n x^{n-1} = x \ln (1-x)
\]

Can even get power series reps'n for functions we can't otherwise write:

\[
\int e^{-x^2} \, dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (2n+1)} x^{2n+1} + C
\]

Can't be written in terms of elem. func'tns

**Taylor series**: if \( f(x) \) has a power series reps'n @ \( x=a \), it's given by its Taylor series:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]
E.g. if $f(x) = \sin(ax)$

- $f(0) = \sin(0) = 0$
- $f'(0) = \cos(0) = 1$
- $f''(0) = -\sin(0) = 0$
- $f'''(0) = -\cos(0) = -1$

So Taylor series @ $a = 0$:

$$\sin(x) = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 - \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \quad \text{(converges for all } x \text{ to } \sin(x))$$

Power series rep's useful for estimating functions $f(x)$ by polynomials.

If $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ on $(a-R, a+R)$

the $N$th Taylor polynomial is:

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N$$

We have: $f(x) \approx T_N(x)$ on $(a-R, a+R)$

where $\approx$ gets better the larger the $N$ and the closer $x$ is to $a$. 
Error given by: \( R_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \) \( \tag{214} \)

Can sometimes bound \( R_N(x) \) w/ alt. series estimation then.

E.g. \( \sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \)

by \( \omega \) alt. series est. then

\( x - \frac{x^3}{3!} + \frac{x^5}{5!} \) is w/in \( \frac{1}{7!} \) of \( \sin(x) \).

So e.g. in \((-1,1)\) this poly approx. \( \sin(x) \) to w/in \( \frac{1}{7!} \).

If alt. series doesn't apply, use Taylor's theorem to bound \( R_N(x) \).