For this series, \( a_n = \frac{1}{n!} x^n \)

So:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x(n+1)!}{(n+1)!} \cdot \frac{2^n}{x^n} \right|
\]

\[
= \lim_{n \to \infty} \frac{n!}{(n+1)!} |x|
\]

\[
= \lim_{n \to \infty} \frac{1}{n+1} |x|
\]

\[
= 0
\]

no matter what \( x \) is!

by ratio test: this power series converges for all real numbers \( x \)!

ex: For which \( x \) does the power series \( \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \) converge?

So:\( \sum \) for this series

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right|
\]

\[
= \lim_{n \to \infty} (n+1) |x|
\]
= \infty \text{ if } x \neq 0.
= 0 \text{ if } x = 0

By ratio test, this series diverges unless \( x = 0 \).

- The power series we’ve seen so far are sometimes called power series centered at 0.

- Can more generally consider expressions of the form:

\[
\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots
\]

where \( a \) is some real number.

- Called the power series centered at \( a \).

\text{Ex: For which value of } x \text{ does the series } \sum_{n=1}^{\infty} \frac{1}{n} (x-3)^n \text{ converge?}

\[
(x-3) + \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 + \ldots
\]
In this case: \( a_n = \frac{1}{n} (x-3)^n \)

we check:

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-3)^{n+1}/n+1}{(x-3)^n/n} \right|
\]

\[
= \lim_{n \to \infty} \left| (x-3) \frac{n}{n+1} \right|
\]

\[
= |x-3|
\]

by ratio test, series converges if

\( |x-3| < 1 \), i.e. \(-1 < x-3 < 1\)

i.e. \(2 < x < 4\).

also, series diverges if

\( |x-3| > 1\), i.e. if \(x > 4\) or \(x < 2\).

What if \(x = 2\) or \(x = 4\)? Ratio test inconclusive, have to check by hand.

If \(x = 4\), series becomes

\[
\sum_{n=1}^{\infty} \frac{1}{n} \to \text{diverges}
\]
If \( x = 2 \), series becomes:
\[
\sum_{n=1}^{\infty} (-1)^n x^n
\]
Converges by dlt series test.
so overall: series converges if \( x \in (-1, 1) \)
and diverges otherwise.

\[\text{Notice: in our examples so far, the interval of x's for which we have convergence has the same center as the series (either 0, or 3) }\]
This turns out to be true in general:

\[\text{Theorem: For a power series } \sum_{n=0}^{\infty} c_n (x-a)^n \text{ exactly one of the following holds:} \]

(i) series converges only if \( x = a \)
(ii) series converges for all \( x \)
(iii) there is a positive real number \( R > 0 \) such that series converges if \( |x-a| < R \) and diverges if \( |x-a| > R \).

See book for proof.
(iii) Says: $x$ in $(c-R, c+R)$

$\Rightarrow$ converge

in $(-\infty, c-R)$ or $(c+R, \infty)$

$\Rightarrow$ diverge

If $x = c-R$ or $c+R$

can converge or diverge

have to check!

$R$ is called the radius of convergence.

- In case (iii), 4 possibilities for the interval of convergence: $(c-R, c+R)$, $(c-f, c+R)$, $(c-R, c+f)$, $(c-R, c+R)$.

- In case (ii) say radius is $\infty$.

Interval is $(-\infty, \infty)$.

- In case (i) say radius is 0.

Interval is $x = c$.

So for:

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + x^2 + \ldots$

radius

interval

$\frac{1}{1-x}$

$(-1, 1)$

$x = 0$

$\infty$

$(-\infty, \infty)$

$1$

$(0, 2)$
ex: Determine radius and interval of convergence.

For: \( \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} \)

**Solution:** Here \( a_n = \frac{(-3)^n x^n}{\sqrt{n+1}} \)

We check: \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-3)^{n+1} x^{n+1}/\sqrt{n+2}}{(-3)^n x^n/\sqrt{n+1}} \right| \)

\[ = \lim_{n \to \infty} 3 |x| \sqrt{\frac{n+2}{n+1}} \]

\[ = 3 |x| \]

By ratio test: if \( 3 |x| < 1 \), converges.

i.e. \( |x| < \frac{1}{3} \)

So radius is \( \frac{1}{3} \).

**What about endpoints?**

If \( x = \frac{1}{3} \), series becomes

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \]

Converges by alt. series test since

\[ \frac{1}{\sqrt{n+1}} \to 0 \] and decreases.
If \( x = -\frac{1}{3} \):

\[
\infty \sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{n+1}
\]

\[
= \infty \sum_{n=0}^{\infty} \frac{(1)^n}{n+1} = \infty \sum_{n=0}^{\infty} \frac{1}{n+1}
\]

Diverges (do lim comparison \( \frac{w}{\frac{1}{n}} \))

So: interval of convergence is \((-\frac{1}{3}, \frac{1}{3}]\)

ex: Find rad. int. of conv. for

\[
\sum_{n=0}^{\infty} \frac{n}{3n+1} (x+2)^n
\]

Sol'n: Here \( a_n = \frac{n(x+2)^n}{3n+1} \)

we check: \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)(x+2)^{n+1}}{3n+2} \cdot \frac{3n+1}{n(x+2)^n} \right| \)

\[
= \lim_{n \to \infty} \left| (x+2) \cdot \frac{n+1}{n} \cdot \frac{1}{3} \right| = \frac{1}{3} |x+2|
\]
So series converges if:
\[
\frac{1}{3} |x+2| < 1
\Rightarrow |x+2| < 3
\Rightarrow -3 < x+2 < 3
\Rightarrow -5 < x < 1
\]

If $x = 5$, series is:
\[
\sum_{n=0}^{\infty} \frac{n (-3)^n}{3^{n+1}}
\]
\[
= \sum_{n=0}^{\infty} \frac{n}{3} \frac{(-3)^n}{3^n}
= \sum_{n=0}^{\infty} \frac{n}{3} (-1)^n
\]
diverges by divergence test.

If $x = 1$, series is:
\[
\sum_{n=0}^{\infty} \frac{n 3^n}{3^{n+1}}
= \sum_{n=0}^{\infty} \frac{1}{3} n
\] also divergent.

so: radius is $3$, interval is $(-5, 1)$