\[
= \left( \frac{n+1}{n} \right)^2 \cdot \frac{1}{3}
\]

hence \[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^2 \frac{1}{3} = \frac{1}{3} < 1. \]

by ratio test, \( \sum (-1)^n \frac{n^3}{2^n} \) converges absolutely (hence converges).

Note: if we’d just wanted to show convergence, could’ve gotten away w/ alt. series test.

(2) What about \( \sum \frac{n!}{100^n} \)?

\[
\text{Sol'n in this case:} \quad \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!}{n!} \cdot \frac{100^n}{100^{n+1}} \right| = \left| \frac{n+1}{100} \right|
\]

so \[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n+1}{100} = \infty \]

by ratio test, series diverges.
(3) What about \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2} \)?

\[
\text{Sol}: \quad \left| \frac{\text{ant} \frac{1}{n}}{an} \right| = \left| \frac{\sqrt{n+1}}{\sqrt{n}} \right| \cdot \frac{1+n^2}{1+(n+1)^2} \\
\quad = \sqrt{\frac{n+1}{n}} \cdot \frac{1+n^2}{n^2+2n+2} \\
\text{So } \lim_{n \to \infty} \left| \frac{\text{ant} \frac{1}{n}}{an} \right| = \lim_{n \to \infty} \frac{\sqrt{n+1}}{n} \cdot \frac{1+n^2}{n^2+2n+2} = 1 \cdot 1 \\
\text{D ratio test inconclusive.}
\]

but observe:

\[
\frac{\sqrt{n}}{1+n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}
\]

and \( \sum \frac{1}{n^{3/2}} \) converges.

By Comparison:

\( \sum \frac{\sqrt{n}}{1+n^2} \) converges.
Thm (Root Test): 

\[ \text{If } \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1 \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges absolutely.} \]

\[ \text{If } \lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1 \text{ or DNE, then } \sum_{n=1}^{\infty} a_n \text{ diverges.} \]

\[ \text{If } \lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \text{ or DNE, inconclusive.} \]

See book for proof (also uses geo. series).

Useful for dealing w/ series involving powers.

Ex: Does \[ \sum_{n=1}^{\infty} \left( \frac{5n+6}{7n+20} \right)^n \] converge?

Sol/n in this case

\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{5n+6}{7n+20} = \frac{5}{7} < 1 \]

\[ \Rightarrow \text{series converges absolutely.} \]
Power Series

Defn let x be a variable. A power series is an expression of the form
\[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots \]
where the \( c_n \)'s are real numbers (called the coefficients of the power series).

Ex: (i) if \( c_n = 1 \) for every \( n \) we get the power series:
\[ \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \]

(ii) if \( c_n = 2^n \) for every \( n \):
\[ \sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \ldots \]

We can think of a power series as an "infinite polynomial."
Only once we specify \( \infty \) does a power series become a series in our original sense. Once we specify an \( x \), we can ask about convergence/divergence of the series (for that \( x \)).

Ex: Consider the series from before: \( \sum x^n = 1 + x + x^2 + x^3 + \ldots \).

If \( x = \frac{1}{2} \), series becomes:

\[
\sum \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots
\]

\[
= 1 + \sum \left(\frac{1}{2}\right)^n \quad \text{geometric series converging to 1}
\]

\[
= 1 + 1 = 2
\]

But if \( x = 1 \) series becomes:

\[
\sum (1)^n = 1 + 1 + 1 + \ldots
\]

which diverges.
Notice a power series \( \sum_{n=0}^{\infty} c_n x^n \) always converges if \( x = 0 \), since in this case:
\[
\sum_{n=0}^{\infty} c_n 0^n = c_0 + 0 + 0 + \ldots = c_0
\]

The question: given a power series \( \sum_{n=0}^{\infty} c_n x^n \), for which \( x \)'s (besides \( c \)) does the series converge?

Ratio and root tests can often help to answer!

**Ex:** for our series:
\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots
\]

If we imagine plugging in for \( x \), so they become an actual series, we have \( c_n = x^n \), so:
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \to \infty} |x| = |x|
\]
So: if $|x| < 1$, i.e. $-1 < x < 1$, series converge by ratio test.

If $|x| > 1$, i.e. $x > 1$ or $x < -1$, series diverge by ratio test.

If $x = 1$, series is:

$$
\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + \ldots \text{ which diverges}
$$

If $x = -1$, series is:

$$
\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots \text{ which diverges}
$$

We've shown: $\sum x^n$ converges if and only if $-1 < x < 1$.

Ex: For which $x$ does the series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converge?

"$1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots$"