11.3 The Integral Test

We know:
\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{converges} \quad \text{(easily = 1)}
\]

also:
\[
\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{diverges}
\]

What about
\[
\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \ ?
\]

An issue: there's no nice formula for the partial sum \( s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \)
so, hard to find
\[
\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

Can we at least determine whether
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges or diverges?}
\]
Can visualize terms in this series as areas of rectangles:

\[
y = \frac{1}{x^2}
\]

But look: can bound total area of rectangles (except the first) by area under \( \frac{1}{x^2} \)!

Precisely:

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}
\]

\[
\leq 1 + \int_{1}^{\infty} \frac{1}{x^2} \, dx
\]

we showed \( = 1 \) earlier

\[
= 1 + 1 = 2
\]
So: using an improper integral we've shown: $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$.

In particular, series converges!

We have not computed exact value
(in fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$)

Could we also use an integral to show a series diverges?

Consider: $1 + \frac{1}{1^2} + \frac{1}{2^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$

Can visualize:

\[ y = \frac{1}{x} \]
Observe! Difference in how we draw rectangles relative to $x$-axis.

Hence: $$\lim_{n \to \infty} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \infty$$

sum of rectangles

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ must diverge as well!

Observations in these examples are formalized in the following:

Theorem (Integral Test): A series $\sum_{n=1}^{\infty} a_n$ converges if and only if $f(x)$ is a continuous, decreasing function (e.g., $\frac{1}{x}, \frac{1}{x^2}$) and $f(x) \geq 0$.

Then: $$\int_{1}^{\infty} f(x) \, dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_{1}^{\infty} f(x) \, dx$$
In particular:

\[ \sum_{n=1}^{\infty} f(n) \text{ converges if and only if } \int_{1}^{\infty} f(x) \, dx \text{ converges.} \]

**Ex:** Determine whether the following series converge.

1) \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = 1 + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \ldots \]

2) \[ \sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4} = \frac{1}{4} + \frac{2}{20} + \ldots \]

**Sol:** 1) \[ \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \sum_{n=1}^{\infty} f(n) \text{ where } f(x) = \frac{1}{x^{1/2}} = \frac{1}{x^{3/2}} \]

Observe: \( \frac{1}{x^{3/2}} \) is cts, decreasing, nonnegative

Also: \[ \int_{1}^{\infty} \frac{1}{x^{3/2}} \, dx = \lim_{t \to \infty} \int_{1}^{t} x^{-3/2} \, dx \]

\[ = \lim_{t \to \infty} \left[ -2x^{-1/2} \right]^{t}_{1} \]

\[ = \lim_{t \to \infty} \left( -\frac{2}{\sqrt{t}} + 2 \right) \]

\[ = 2 \]
So \( \int_{1}^{\infty} x^{-\frac{3}{2}} \, dx \) converges.

By integral test: \( \sum_{n=1}^{\infty} n^{-3/2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) converges too!

2) \( \sum_{n=1}^{\infty} \frac{n^3}{n^4 + 1} = \sum_{n=1}^{\infty} f(x) \) where \( f(x) = \frac{x^3}{x^4 + 1} \)

we have: \( \int_{1}^{\infty} \frac{x^3}{x^4 + 1} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{x^3}{x^4 + 1} \, dx \)

\( u = x^4 + 1 \)
\( du = 4x^3 \, dx \)

\( \int \frac{1}{4} u^{-1} \, du \)
\( = \frac{1}{4} \ln |u| \)
\( = \frac{1}{4} \left( \ln |x^4 + 1| \right) \)

\( = \lim_{t \to \infty} \frac{1}{4} \ln |x^4 + 1| \)
\( = \lim_{t \to \infty} \left( \frac{1}{4} \ln (t^4 + 1) - \frac{1}{4} \ln (s) \right) \)

So integral diverges = \( \infty \)

\( \Rightarrow \sum_{n=1}^{\infty} \frac{n^3}{n^4 + 1} \) diverges too!
Estimating $\sum_{n=1}^{\infty} a_n$

Given a series of the form
$$\sum_{n=1}^{\infty} f(n)$$
where it decreases functionally

that we can't evaluate exactly (e.g., $\sum \frac{1}{n^2}$), we'd like to be able to estimate it well.

The $n$th partial sum $S_n = a_1 + a_2 + \ldots + a_n$

=gives such an estimate.

-the larger the $n$, the better $S_n$
approximates $\sum_{n=1}^{\infty} a_n$

-can we bound the error?

Define: the $n$th remainder

$$R_n = \sum_{n=1}^{\infty} a_n - S_n$$

= $a_{n+1} + a_{n+2} + \ldots$

Can estimate $R_n$ with an integral:

$$y = f(x)$$

$$\text{area} = \int_{n}^{\infty} f(x) \, dx$$

Sum of rectangles' areas:

$$a_{n+1} + a_{n+2} + \ldots = R_n$$

$$\leq \int_{n}^{\infty} f(x) \, dx$$

On the other hand:

$$c_{n+1} = f(n+1)$$
$$c_{n+2} = f(n+2)$$
$$c_{n+3} = f(n+3)$$

$$\text{area} = \int_{n+1}^{\infty} f(x) \, dx$$
sum of rectangles' areas = \( a_{n+1} + a_{n+2} + \ldots \)

\[ = R_n \text{ again} \]

\[ \geq \int_{n+1}^{\infty} f(x) \, dx \]

We've shown:

**Theorem:** If \( f(x) \) is \( \leq 0 \), decreasing, nonnegative and \( R_n = \sum_{k=1}^{n} f(k) \),

\[ = a_{n+1} + a_{n+2} + \ldots \]

Then:

\[ \int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx \] (A)

assuming the series \( \sum_{k=1}^{\infty} f(k) \) converges.

Another way of writing this inequality:

\[ S_n + \int_{n+1}^{\infty} f(x) \, dx \leq \sum_{k=1}^{\infty} f(k) \leq S_n + \int_{n}^{\infty} f(x) \, dx \] (*)&

**ex:** Consider \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

a) Compute \( S_{10} \), estimate \( R_{10} \)

b) How large does \( n \) need to be to guarantee \( R_n < 0.0005 \)?