(5) we have that $\frac{1-x}{e^x} \leq 0$ if $x \geq 1$ (141)

So x/e^x is decreasing on $[1,\infty)$

$\Rightarrow n/e^n = ne^{-n}$ is too, i.e.

(1) $n+1e^{-(n+1)} \leq ne^{-n}$ for all $n \geq 1$

Furthermore $\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n}$

$= \lim_{x\to\infty} \frac{x}{e^x}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$. (2)

So (1) + (2) $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} ne^{-n}$ converges, by thm.

Estimates for alternating series.

$\hookrightarrow$ we know how to estimate series w/ integrals.

if $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ our series, where $f(x)$ decreasing

then S_n is our estimate, $R_n = \sum a_n - S_n$ our error, we have

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

(6)

↳ but integrals are hard!

turns out: for alternating series (w/ decreasing terms), R_n is easy to estimate

— no integrals required!

Theorem: If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is an alternating series s.t.

already knew $\sum (-1)^{n-1} b_n$ converges

(1) $b_{n+1} \le b_n$ for all n

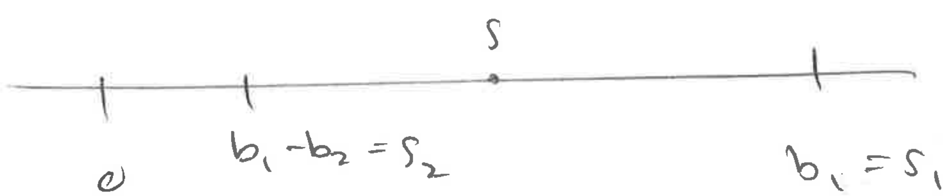
(2) $\lim_{n \to \infty} b_n = 0$

Then: $|R_n| \le b_{n+1}$

i.e. $-b_{n+1} \le R_n \le b_{n+1}$.

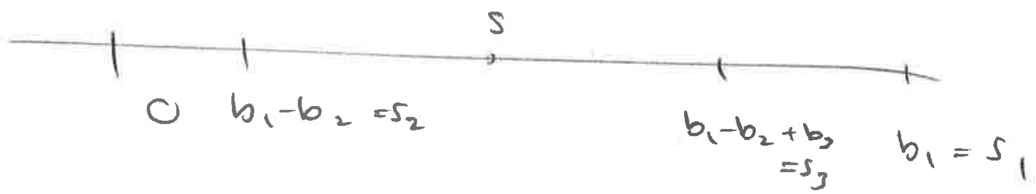
Proof by (some) picture

Let $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$. Then $|R_n| = |s - s_n|$ is the positive distance from s to s_n



distance from s to s_1 $\le$ distance from s_2 to s_1

i.e. $|R_1| \le b_2$



distance from s to s_2 $\le$ distance from s_3 to s_2

i.e. $|R_2| \le b_3$

and so on!

in general s bdd between s_n and s_{n+1} so that:

distance from s to s_n $\le$ distance from s_{n+1} to s_n

i.e. $|R_n| \le |s_{n+1} - s_n| = b_n$ ✓

8. ex. Consider the alt. harmonic series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n}\right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots$$

→ use S_{10} as an estimate for the sum.

→ bound the error in this estimate (i.e. bound R_{10})

→ how many terms to guarantee $R_n \leq .001$?

Sol'n: → $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{10} = .6456\ldots = S_{10}$

→ $R_{10} \leq b_{11} = \frac{1}{11} = .0909\ldots$

→ want: $R_n \leq .001$

Know $R_n \leq \frac{1}{n+1}$ so: want: $\frac{1}{n+1} \leq .001 = \frac{1}{1000}$

$\Rightarrow n+1 \geq 1000$

$\Rightarrow n \geq 999$ ✓

(in fact $\sum (-1)^{n-1} \frac{1}{n} = \ln(2) = .693147\ldots$

①

Ratio and Root tests

— tests for convergence / divergence so far.

Test	Applies to
Divergence test	all series
Geometric series test	geometric series $\sum_{n=1}^{\infty} ar^{n-1}$
p-series test	p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$
Integral test	series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ where $f(x) > 0$ and decreasing on $[1, \infty)$
Comparison tests	series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$
Limit comparison test	series $\sum_{n=1}^{\infty} a_n$ w/ $a_n \geq 0$.
Alternating series test	alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$

— two new tests today: ratio and root tests

- both powerful — based on geo. series test.

② first need:

<u>Def'n</u> • A series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

• A series $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u> if $\sum a_n$ converges but $\sum |a_n|$ diverges.

<u>Ex's</u>: • $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n^2}\right) = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$

is absolutely convergent since $\left|(-1)^{n-1}\left(\frac{1}{n^2}\right)\right| = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges by p-series test.

• $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

is conditionally convergent since it converges but $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)$ does not.

• $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$

diverges by divergence test (neither abs. nor cond. convergent).

(3) <u>Theorem</u>: if $\sum a_n$ is absolutely convergent, then it is convergent.

<u>Pf</u>: - Suppose $\sum_{n=1}^{\infty} |a_n|$ converges.

- observe: $0 \leq a_n + |a_n| \leq 2|a_n|$ for every n.
- and since $\sum |a_n|$ converges, we know $\sum 2|a_n| = 2\sum |a_n|$ converges too.
- thus, by comparison: $\sum a_n + |a_n|$ converges.
- now we can apply our summation rules:

$\sum |a_n|$ $\sum a_n + |a_n|$ both convergent

$$\Rightarrow$$

$$\sum a_n + |a_n| - \sum |a_n| = \sum a_n \text{ converges}$$

<u>ex</u>: Show $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2+e^n)}{n^3}$ converges.

<u>Sol'n</u>: - series has both + and - terms, but isn't alternating.

- but notice $\left|\frac{\sin(n^2+e^n)}{n^3}\right| \leq \frac{1}{n^3}$

- So by Comparison

$$\sum \left| \frac{\sin(n^2+e^n)}{n^3} \right| \quad \text{converges.}$$

- hence by theorem $\sum_{n=1}^{\infty} \frac{\sin(n^2+e^n)}{n^3}$ conv.

<u>Note</u>: Converse to theorem is <u>not</u> true.

e.g. $\sum (-1)^{n-1} \frac{1}{n}$ converges but is <u>not</u> absolutely convergent. (Thm justifies terminology "conditionally convergent")

Onto our new tests:

<u>Theorem</u> (Ratio test). Sps $\sum_{n=1}^{\infty} a_n$ is a series.

(i) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum a_n$ is <u>absolutely</u> <u>convergent</u> (hence convergent)

(ii) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then $\sum a_n$ is <u>divergent</u>. (including if $L = \infty$)

(iii) if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ or DNE, test is inconclusive.

Pf (sketch): (i) Sps $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$.

We'll prove $\sum |a_n|$ converges.

- There is an integer N, and real number r with $L < r < 1$ such that:

  for every $n \geq N$, $\left|\frac{a_{n+1}}{a_n}\right| \leq r$

  i.e. $|a_{n+1}| \leq r|a_n|$

- So in particular:

  → $|a_{N+1}| \leq r|a_N|$

  → and $|a_{N+2}| \leq r|a_{N+1}| \leq r^2|a_N|$

  → and $|a_{N+3}| \leq r^3|a_N|$ ... etc.

  in general: $|a_{N+k}| \leq r^k|a_N|$

- So if we consider the tail sum of $\sum_{n=N}^{\infty} |a_n|$ beginning at N, by comparison we have:

$$\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_N| r^{n-1}$$

← geo series converges to $\frac{|a_N|}{1-r}$.

since $r<1$

(6)

- Hence tail-sum

$$\sum_{n=N}^{\infty} |a_n| \quad \text{converges}$$

Hence entire sum

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_{N-1}| + \sum_{n=N}^{\infty} |a_n|$$

converges too ✓

(ii) Same idea, but instead bound $|a_{N+k}|$ below by $r^k |a_N|$ with $r > 1$. Apply divergence test.

(iii) if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, can't use geo. series to compare, so test inconclusive.

Ex's : (1) Does $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converge absolutely?

Sol'n : $$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{(-1)^{n+1} (n+1)^3 / 3^{n+1}}{(-1)^n (n)^3 / 3^n} \right|$$

$$= \left| \frac{(-1)^{n+1} (n+1)^3 \cdot 3^n}{(-1)^n n^3 \, 3^{n+1}} \right| = \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}$$

$$\text{hence: } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{1}{3}$$

$$= \frac{1}{3} < 1$$

$\to$ by ratio test, $\sum (-1)^n \frac{n^3}{3^n}$ converges absolutely (and hence converges).

Note: if we'd just wanted to show convergence could have gotten away w/ alternating series test..

(2) What about $\sum_{n=1}^{\infty} \frac{n!}{100^n}$?

Sol'n: in this case:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!} \cdot \frac{100^n}{100^{n+1}}\right|$$

$$= \left|\frac{n+1}{100}\right| = \frac{n+1}{100}$$

8

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{100} = \infty$

by ratio test, series diverges.

(3) What about $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$?

Sol'n: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2}\right|$

$$= \sqrt{\frac{n+1}{n}} \cdot \frac{1+n^2}{2+2n+n^2}$$

So $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \frac{n^2+1}{n^2+2n+2}$

$$= 1 \cdot 1 = 1$$

$\Rightarrow$ ratio test inconclusive.

but observe:

$\sum \frac{1}{n^{3/2}}$ converges

$$\frac{\sqrt{n}}{1+n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

So by comparison: $\sum \frac{\sqrt{n}}{1+n^2}$ converges ✓

Thm (Root test) Sps $\sum_{n=1}^{\infty} a_n$ is a series.

(i) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ converges absolutely

(ii) if $\lim_{n\to\infty} \sqrt{|a_n|} = L > 1$, then $\sum a_n$ diverges (includes if $L = \infty$)

(iii) if $\lim_{n\to\infty} \sqrt{|a_n|} = 1$ or DNE, inconclusive.

→ won't prove, but also uses comparison to geometric series

→ useful for dealing w/ series involving nth powers in a_n.

ex: Does $\sum_{n=1}^{\infty} \left(\frac{5n+6}{7n+20}\right)^n$ converge?

Sol'n in this case

$$\sqrt[n]{|a_n|} = \frac{5n+6}{7n+20}$$

So $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{5n+6}{7n+20} = \frac{5}{7} < 1$.

→ by root test, series converges absolutely ✓

Power Series

Def'n Let x be a variable. A power series is an expression of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where the c_n's are real numbers (called the coefficients of the power series).

(note: for power series the sum begins at $n=0$ by convention)

ex ① if $c_n = 1$ for every n we get the power series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

② if $c_n = 2^n$ for every n we get

$$\sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

②

- can think of a power series as an "infinite polynomial"
- only once we specify x does a power series become a series in our previous sense.
- once we specify an x, can ask about convergence and divergence of the series (for that x).

ex: ① Consider our series from before:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

if $x = 1/2$ this power series becomes

$$\sum_{n=0}^{\infty} (1/2)^n = 1 + 1/2 + 1/4 + 1/8 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} (1/2)^n$$

geo series converging to 1

$$= 1 + 1 = 2.$$

but if $x = 1$ the power series becomes

$$\sum_{n=0}^{\infty} (1)^n = 1 + 1 + 1 + \cdots$$

which diverges.

Notice: a power series $\sum_{n=0}^{\infty} c_n x^n$ always converges when $x = 0$, since in this case series becomes

$$\sum_{n=0}^{\infty} c_n 0^n = c_0 + 0 + 0 + 0 + \cdots = c_0$$

The big Q: for a given power series $\sum_{n=0}^{\infty} c_n x^n$, for which x's (besides 0) does the series converge?

↳ ratio and root tests v useful in answering big Q in many cases.

ex: for our series:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

if $x = r$, series becomes

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$$

For this series, $a_n = r^n$ so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{r^{n+1}}{r^n} \right| = \lim_{n\to\infty} |r|$$

(4)

$= |r|$

thus: - if $|r|<1$, i.e. $-1<r<1$, the series converges by ratio test.

- if $|r|>1$, i.e. $r<-1$ or $r>1$, then series diverges by ratio test.

if $r=1$, series is:

$$\sum_{n=0}^{\infty} 1^n = 1+1+1+\cdots \quad \text{which diverges}$$

$r=-1$ series is:

$$\sum_{n=0}^{\infty} (-1)^n = 1-1+1-1+\cdots \quad \text{which diverges.}$$

we've shown $\sum_{n=0}^{\infty} x^n$ converges if and only if $-1<x<1$. ✓

ex: For which x does the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

converge?

(5) for this series, $a_n = \frac{1}{n!} x^n$

so: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{1/(n+1)! \; x^{n+1}}{1/n! \; x^n} \right|$

$= \lim_{n \to \infty} \frac{n!}{(n+1)!} \cdot |x|$

$= \lim_{n \to \infty} \frac{1}{n+1} |x|$

$= 0$

no matter what x is!

by ratio test: this series converges for all real numbers x.

ex: For which x does the power series $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \ldots$ converge?

Sol'n: for this series:

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)! \, x^{n+1}}{n! \, x^n} \right|$

$= \lim_{n \to \infty} (n+1) |x|$

6

$= \infty \quad \text{if} \quad x \neq 0$

$= 0 \quad \text{if} \quad x = 0$

Hence: by ratio test, this series diverges unless $x = 0$.

- the power series we've seen so far are sometimes called <u>power series centered at 0</u>.

- can more generally consider expressions of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a is a real number.

- called a <u>power series centered at a</u>.

<u>Ex</u>: For which values of x does the series $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ converge? $\longrightarrow$

$= (x-3) + (x-3)^2/2 + (x-3)^3/3 + \cdots$

can view this as power series centered at 3 w/ $c_0 = 0$ and $c_n = \frac{1}{n}$ for n

<u>Sol'n</u>: in this case $a_n = \frac{1}{n}(x-3)^n$

~~[illegible]~~: we check:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}/n+1}{(x-3)^n/n}\right|$$

$$= \lim_{n\to\infty}\left|(x-3)\frac{n}{n+1}\right|$$

$$= \lim_{n\to\infty} |x-3| \frac{n}{n+1} \to 1$$

(ii)

$= |x-3|$

by ratio test, if $|x-3|<1$, series converges

i.e. if $-1 < x-3 < 1$

$\Rightarrow \quad 2 < x < 4$

also by ratio test, if $|x-3|>1$, series diverges, i.e. if $x>4$ or $x<2$.

What about if $x=2$ or $x=4$?

Ratio test gives no info: have to check by hand.

if $x=4$ series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n}, \text{ diverges}$$

if $x=2$ series becomes

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

$\rightarrow$ converges by alt. series test

So: Series converges for x in $[-2,4)$ ✓

(iii) notice: for all examples of power series so far (both those centered at 0 and at 3), the interval of x's for which series converges has same center as series (either 0 or 3, in our ex's).

Turns out to be true in general.

Theorem For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ exactly one of the following holds:

(i) the series converges only for $x=a$
(ii) the series converges for all x
(iii) there is a positive real number $R>0$ s.t. the series converges if $|x-a|<R$, and diverges if $|x-a|>R$

(iii) says if x in $(a-R, a+R) \to$ converge
if x in $(-\infty, a-R)$ or $(a+R, \infty) \to$ diverge

if $x=a-R$ or $x=a+R$ can have convergence or divergence depending on case.

(iv) - R is called <u>radius</u> <u>of</u> <u>convergence</u>

- In case (iii), four possibilities for

<u>interval</u> <u>of</u> <u>convergence</u>: $(a-R, a+R)$

$[a-R, a+R)$

$(a-R, a+R]$

$[a-R, a+R]$

- In case (ii) say: <u>radius</u> <u>of</u> <u>conv</u> is ∞

<u>interval</u> <u>of</u> <u>conv</u> is $(-\infty, \infty)$

- in case (i): radius is 0.

e.g. for our examples so far:

	<u>radius of conv</u>	<u>interval</u>
$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$	1	$(-1, 1)$
$\sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + \cdots$	0	$x = 0$
$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	∞	$(-\infty, \infty)$
$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	1	$[2, 4)$

v) ex: determine radius + interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Sol'n here: $a_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$

we check: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1} / \sqrt{n+2}}{(-3)^n x^n / \sqrt{n+1}} \right|$

$= \lim_{n\to\infty} 3|x| \frac{\sqrt{n+2}}{\sqrt{n+1}}$

$= \lim_{n\to\infty} 3|x| \sqrt{\frac{n+2}{n+1}} \to 1$

$= 3|x|$

by ratio: if $3|x| < 1$ series converges

i.e. $|x| < 1/3 \Rightarrow x$ in $(-1/3, 1/3)$

So radius is $1/3$

what about endpoints?

if $x = 1/3$ series is: $\sum \frac{(-3)^n (1/3)^n}{\sqrt{n+1}}$

$= \sum \frac{(-1)^n}{\sqrt{n+1}}$ → $\frac{1}{\sqrt{n+1}} \to 0$ + decreasing

converges by alternating series test

vi) if $x = -1/3$ series is $\sum \frac{(-3)^n(-1/3)^n}{\sqrt{n+1}}$

$$= \sum \frac{(1)^n}{\sqrt{n+1}} = \sum \frac{1}{\sqrt{n+1}}$$

$\to$ diverges (do lim comparison w/ $\frac{1}{\sqrt{n}}$)

So interval of convergence is $(-1/3, 1/3]$.

ex. Find the ~~[illegible]~~ radius and interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{n}{3^{n+1}} (x+2)^n$$

Sol'n here $a_n = \frac{n(x+2)^n}{3^{n+1}}$

we check: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \dfrac{\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}}{\frac{n(x+2)^n}{3^{n+1}}} \right|$

$$= \lim_{n\to\infty} \left| (x+2) \overbrace{\frac{n+1}{n}}^{\to 1} \cdot \frac{1}{3} \right|$$

$$= \frac{1}{3}|x+2|$$

vii) Series converges if

$$\frac{1}{3}|x+2| < 1$$

$$\Rightarrow |x+2| < 3$$

i.e. $-3 < x+2 < 3$

$$\Rightarrow -5 < x < 1.$$

if $x = -5$ series is $\sum_{n=0}^{\infty} \frac{n(-3)^n}{(3)^{n+1}}$

$$= \sum n \frac{(-3)^n}{3^n} \frac{1}{3}$$

$$= \sum \frac{n(-1)^n}{3}$$

alternating but divergent.

if $x = 1$ series is $\sum_{n=0}^{\infty} \frac{n\, 3^n}{3^{n+1}}$

$$= \sum_{n=0}^{\infty} \frac{1}{3} n$$

which is also divergent.

$\rightarrow$ radius is 3

interval is $(-5, 1)$

viii) ex: determine ~~o~~ interval of convergence for:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$$

Sol'n: easier to use root test here

we check: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^n}\right|}$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{|x-2|^n}{n^n}}$$

$$= \lim_{n\to\infty} \frac{|x-2|}{n}$$

$= 0$, no matter what x is

by root test, series converges for all x ✓