11.3 The Integral Test

- We've shown:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

converges (actually: $= 1$)

also: $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges

- What about

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} \; ?$$

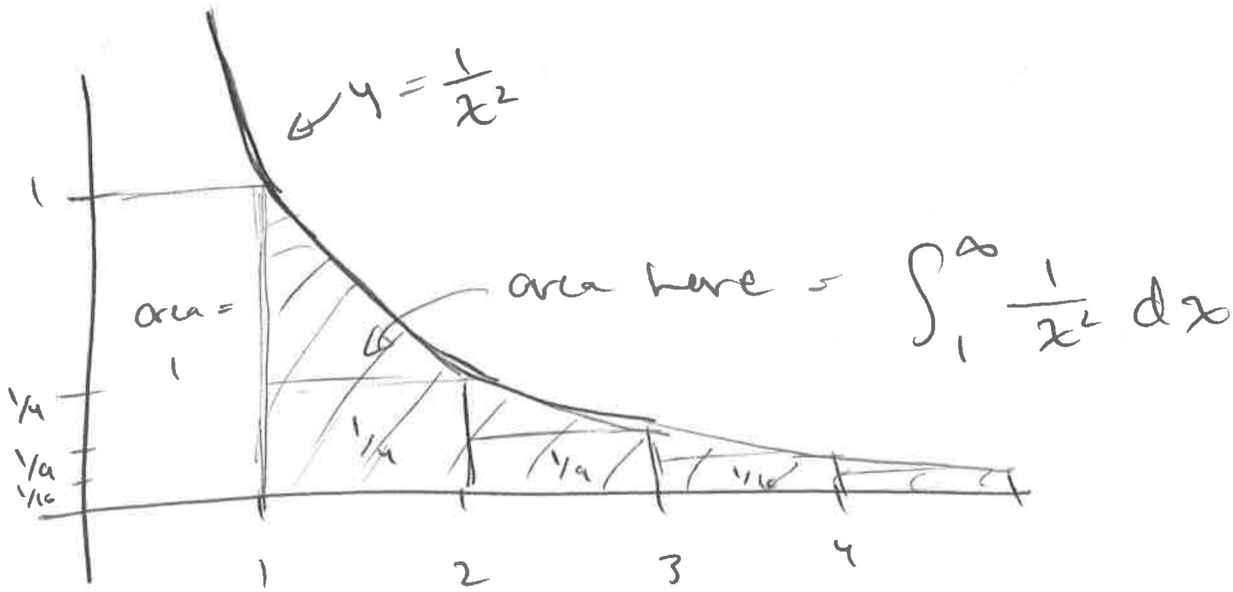
A problem: no nice fmla for nth partial sum $s_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$

therefore, hard to find

$$\lim_{n \to \infty} s_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Q: can we at least determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges?

Can visualize the series w/ some rectangles:



But look: can bound total area of rectangles, except the first, by area under $\frac{1}{x^2}$!

Precisely, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$\leq 1 + \int_1^{\infty} \frac{1}{x^2}\,dx$$

we showed earlier $= 1$

$$= 1 + 1 = 2.$$

So: using an improper integral we've shown

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$$

in particular — it converges!

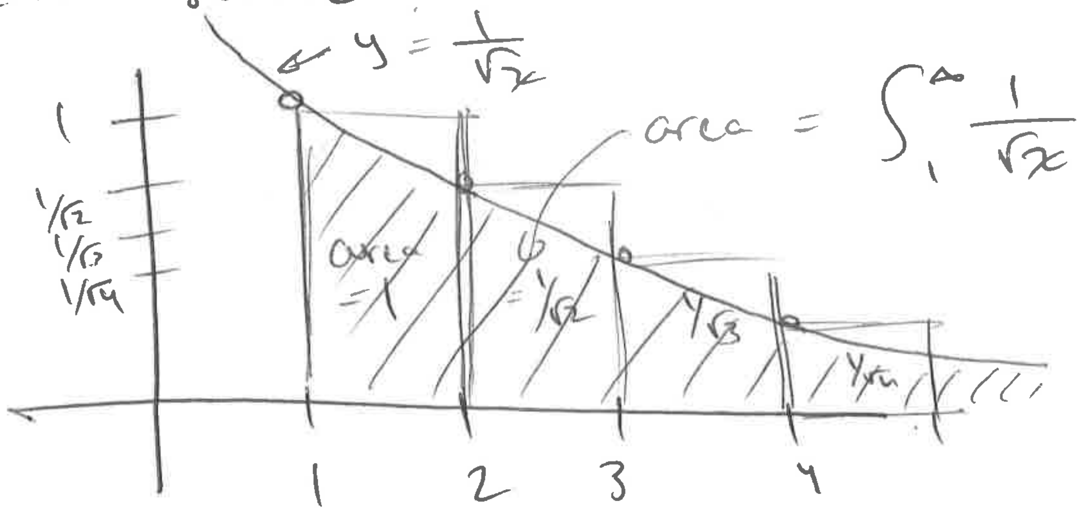
We have <u>not</u> computed exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (in fact $= \pi^2/6$), but we've done a lot.

↳ could we also use an integral to prove a series <u>diverges</u>?

Consider:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Can visualize:



(observe difference in how we draw rectangles relative to x-axis)

Hence $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \infty$ ← diverges!

sum of rectangles

Hence $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ must diverge as well!

Observations in these examples are formalized in following:

Theorem (The Integral Test) Suppose we define a series by $a_n = f(n)$ where $f(x)$ is a continuous decreasing function (e.g. $\frac{1}{x^2}, \frac{1}{\sqrt{x}}$) with $f(x) \geq 0$

then:

$$\int_1^{\infty} f(x)\, dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f(x)\, dx$$

in particular:

$\sum_{n=1}^{\infty} f(n)$ converges if and only if

$\int_1^{\infty} f(x)\,dx$ converges!

ex: determine whether the following series converge:

1) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots$

2) $\sum_{n=1}^{\infty} \frac{n^3}{n^4+4} = \frac{1}{4} + \frac{8}{20} + \cdots$

Sol'n: 1) we have $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} f(n)$

where $f(x) = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$

$$\int_1^{\infty} \frac{1}{x^{3/2}}\,dx = \lim_{t\to\infty} \int_1^t x^{-3/2}\,dx$$

$$= \lim_{t\to\infty} -2x^{-1/2}\Big|_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{2}{\sqrt{t}} + 2\right)$$

$$= 2$$

⑥

So $\int_1^\infty \frac{1}{x^{3/2}}$ converges

$\Rightarrow$ by theorem $\sum_{n=1}^\infty \frac{1}{n^{3/2}} = \sum_{n=1}^\infty \frac{1}{n\sqrt{n}}$

converges too!

2) $\sum_{n=1}^\infty \frac{n^3}{n^4+4} = \sum_{n=1}^\infty f(n)$ where $f(x) = \frac{x^3}{x^4+4}$

So: $\int_1^\infty \frac{x^3}{x^4+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{x^3}{x^4+4}\,dx$

$u = x^4+4$

$du = 4x^3\,dx$

$\int \frac{1}{4} u^{-1}\,du$

$= \frac{1}{4}\ln|u|$

$= \frac{1}{4}\ln|x^4+4|$

$= \lim_{t\to\infty} \frac{1}{4}\ln|x^4+4|\Big|_1^\infty$

$= \lim_{t\to\infty} \left(\frac{1}{4}\ln(t^4+4) - \frac{1}{4}\ln(5)\right)$

$= \infty$

So $\int_1^\infty \frac{x^3}{x^4+4}$ diverges

$\Rightarrow \sum_{n=1}^\infty \frac{n^3}{n^4+4}$ diverges too!

Estimating $\sum_{n=1}^{\infty} a_n$

- Given a series of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n) \text{ on cts decreasing function}$$

that we can't compute explicitly (e.g. $\sum_{n=1}^{\infty} \frac{1}{n^2}$), we'd like to be able to <u>estimate</u> it well.

- The nth partial sum $s_n = a_1 + a_2 + \cdots + a_n$ gives such an estimate.
- the larger the n, the better s_n approximates $\sum_{n=1}^{\infty} a_n$

↳ but can we say how far we're off for a given n?

we define the <u>nth</u> <u>remainder</u>

$$R_n = \sum_{n=1}^{\infty} a_n - s_n$$

$$= a_{n+1} + a_{n+2} + \cdots$$

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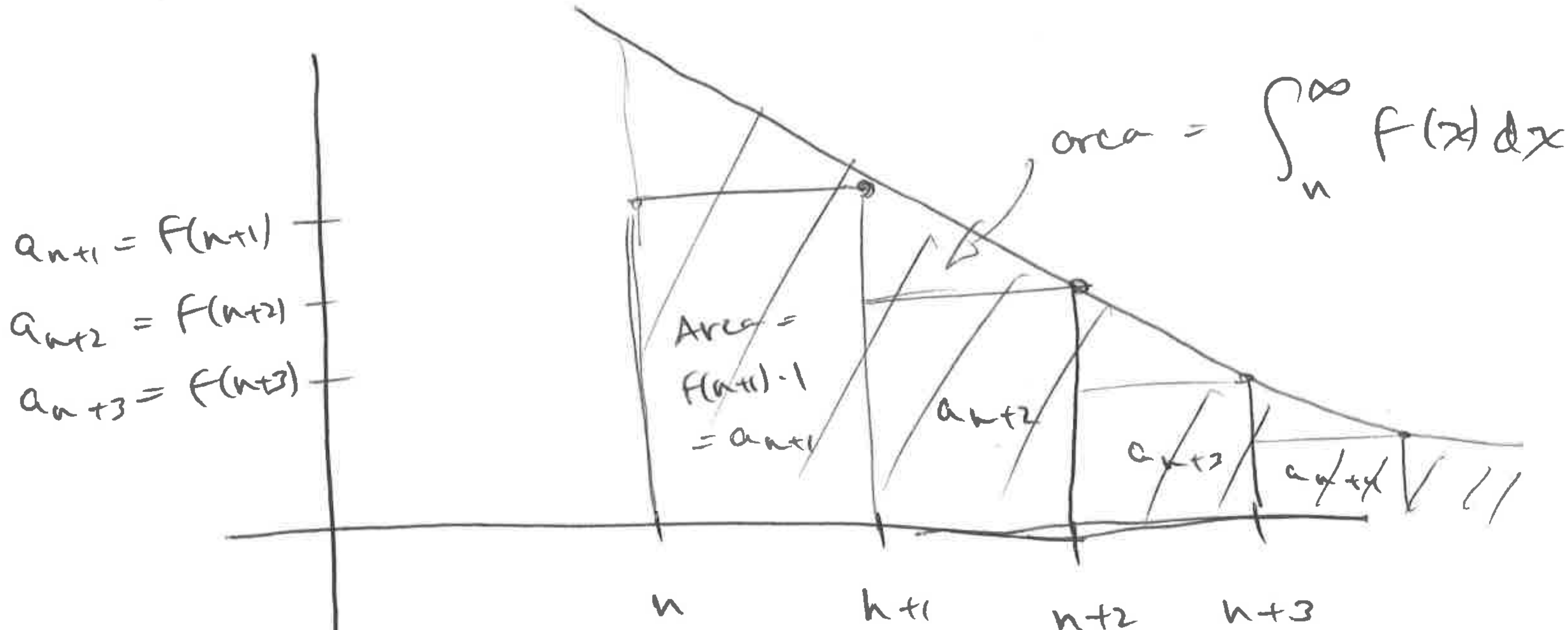
- we can estimate R_n using a similar idea to before:

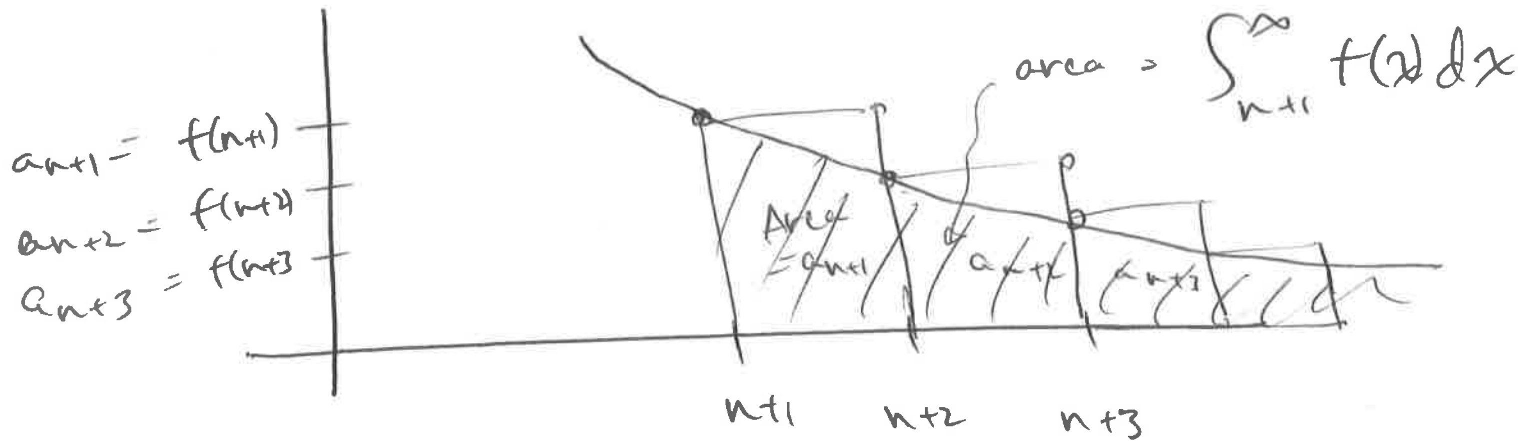


sum of rectangles $= a_{n+1} + a_{n+2} + \cdots$

$$= R_n$$

$$\leq \int_n^\infty f(x)\,dx$$

On the other hand:



so: sum of rectangles $= a_{n+1} + a_{n+2} + \cdots$

$$= R_n$$

$$\geq \int_{n+1}^{\infty} f(x)\,dx$$

overall: Theorem: if $f(x)$ is cts and decreasing and $R_n = \sum_{n=1}^{\infty} f(n) - S_n = a_{n+1} + a_{n+2} + \cdots$, then:

$$(\Delta) \int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

assuming the series $\sum f(n)$ converges

another way of writing this inequality is:

$$(*) \quad S_n + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \leq S_n + \int_{n}^{\infty} f(x)\,dx$$

ex: a) Consider $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Compute S_{10} and estimate R_{10}.

b) How large does n need to be for $R_{10} < .0005$?

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(12)

Sol'n: a) $S_{10} = 1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{1000}$

$= 1.1975$

b) by theorem

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx \le R_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

can check: $\int_{k}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2k^2}$

so we get:

$$\frac{1}{2 \cdot 11^2} \le R_{10} \le \frac{1}{2 \cdot 10^2}$$

$$\underbrace{\frac{1}{242} \le R_{10} \le \frac{1}{200}}$$

this tells us that our "error" R_{10} is $\le .005 = \frac{1}{200}$

i.e. the actual sum

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = S_{10} + R_{10}$$

$$= 1.1975 + R_{10}$$

So: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ w/in .005 of 1.1975.

(1c) Looking at both sides of the inequality actually gives more information:

Can rewrite:

$$\frac{1}{242} \le R_{10} \le \frac{1}{200}$$

$$\Rightarrow \frac{1}{242} \le \left(\sum_{n=1}^{\infty} \frac{1}{n^3}\right) - S_{10} \le \frac{1}{200}$$

$$\Rightarrow S_{10} + \frac{1}{242} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le S_{10} + \frac{1}{200} \quad (*)$$

$$S_{10} + \frac{1}{242} = 1.1975 + \frac{1}{242} = 1.2016\ldots$$

$$S_{10} + \frac{1}{200} = 1.1975 + \frac{1}{200} = 1.2025$$

$$1.2016\ldots \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1.2025$$

better than before!

diff between these values only .00086...

11.4 The comparison tests

First things first: going forward, you may assume the following:

<u>Thm</u> ("p-test")

(1) the integral $\int_1^\infty x^p \, dx$ converges if and only if $p > 1$

(2) the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

↳ we've checked particular cases, but you can now take for granted on HW, etc.

<u>Back to series:</u>

<u>ex</u>: does $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ converge?

– integral test fails us: $\int \frac{1}{2^x+x} \, dx$ not easy to evaluate:

– instead just observe:

$$\frac{1}{2^n+n} < \frac{1}{2^n} \quad \text{for every } n$$

- so we should have:

$$\sum_{n=1}^{\infty} \frac{1}{2^n+n} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

↳ in particular, $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ <u>converges</u>.

<u>ex</u>: what about $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$?

- we already checked w/ integral test that series diverges
- p-test works too
- another way: observe that for every n we have

$$\frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

- so should have:

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\substack{\text{diverges} \\ = \infty}} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}_{\text{so this diverges too.}}$$

- in both examples: <u>comparing</u> an unknown series to a known one to determine convergence or divergence.

More generally we have:

<u>Theorem</u> ("comparison tests")

Suppose $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are series w/ positive terms.

① if $a_n \leq b_n$ for every n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

② if $a_n \geq b_n$ for every n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

our examples above illustrate both types of comparison.

→ these comparison tests are useful, but don't always get us there.

<u>ex</u>: does $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ converge?

— instinct: <u>yes</u> since "looks like" $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

— but direct comparison doesn't work since

- $\frac{1}{2^n} < \frac{1}{2^n - n}$ for every n

$\Rightarrow$ convergence of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ doesn't imply convergence of $\sum_{n=1}^{\infty} \frac{1}{2^n-n}$ directly.

Fortunately we have another comparison test we can use.

Thm ("Limit comparison test") Sps $\sum a_n$ and $\sum b_n$ are series w/ positive terms, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \quad \text{where } 0 < c < \infty$$

then: $\sum_{n=1}^{\infty} a_n$ converges

if and only if

$\sum_{n=1}^{\infty} b_n$ converges.

Pf: $\exists N$ s.t. $\forall n > N$

$\frac{c}{2} \leq \frac{a_n}{b_n} \leq 2c$

$\Rightarrow b_n \frac{c}{2} \leq a_n \leq 2c b_n$

$\Rightarrow \frac{c}{2}\sum_{N}^{\infty} b_n \leq \sum_{N}^{\infty} a_n \leq 2c\sum_{N}^{\infty} b_n$

if $\sum b_n$ div $\Rightarrow \sum a_n$ div

if $\sum b_n$ conv $\Rightarrow \sum a_n$ conv

So: back to our ex:

observe: $\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n-n}} = \lim_{n\to\infty} \frac{2^n - n}{2^n}$

$$= \lim_{x\to\infty} \frac{2^x - x}{2^x}$$

$$= \lim_{x\to\infty} 1 - \frac{x}{2^x} \leftarrow \frac{\infty}{\infty}$$

$$= 1 - \lim_{x\to\infty} \frac{x}{2^x}$$

$$\overset{L'H}{=} 1 - \lim_{x\to\infty} \frac{1}{(\ln 2) 2^x}$$

$$= 1 - 0 = 1 \leftarrow 0 < 1 < \infty$$

So by limit comparison

$\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ converges, since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ does

ex: Does $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ converge? (on positive terms)

- instinct: $\frac{2n^2+3n}{\sqrt{5+n^5}}$ "looks like" $\frac{2n^2}{\sqrt{n^5}}$

$$= \frac{2n^2}{\sqrt{n}\, n^2}$$

$$= \frac{2}{\sqrt{n}}$$

- we know $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges,

So let's ~~compare~~ compare terms:

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$$\lim_{n\to\infty} \frac{\dfrac{2n^2+3n}{\sqrt{5+n^5}}}{\dfrac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{\sqrt{5+n^5}}$$ divide top + bottom by $n^{5/2}$

$$= \lim_{n\to\infty} \frac{2+\cancel{\frac{3}{n}}}{\sqrt{\cancel{\frac{5}{n^5}}+1}}$$

$$= \frac{2}{\sqrt{1}} = 2$$

Since $0 < 2 < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

so must $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ ✓

Estimating series via comparison

We saw before: integrals can help us estimate series by giving us bounds on remainders.

Specifically, we showed: Given a series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$, we have:

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

where $R_n = \sum_{n=1}^{\infty} a_n - S_n = \underbrace{a_{n+1}}_{=f(n+1)} + \underbrace{a_{n+2}}_{=f(n+2)} + \dots$

↳ can use this to estimate errors of series $\sum f(n)$ for which we can't compute $\int f(x)\,dx$ easily.

idea: just compare to an easier integral.

② ex. - Use the sum of the first 10 terms to estimate the series

$$\sum_{n=1}^{\infty} \frac{1}{n^5+5}.$$

- Bound the remainder in this estimate using a comparison.

Sol'n: - In this case,

$$S_{10} = \frac{1}{1^5+5} + \frac{1}{2^5+5} + \cdots + \frac{1}{10^5+5}$$

$$\approx .19926\ldots$$

- we know from above:

$$R_{10} = \sum_{n=1}^{\infty} \frac{1}{n^5+5} - S_{10} \le \int_{10}^{\infty} \frac{1}{x^5+5}\,dx$$

but this is hard to evaluate: would need to factor and do a p.f.d.

(3) (136)

to get an easier estimate observe

$$\frac{1}{x^5+5} \le \frac{1}{x^5} \quad \text{for } x \ge 0$$

Hence $\int_{10}^{\infty} \frac{1}{x^5+5}\,dx \le \int_{10}^{\infty} \frac{1}{x^5}\,dx$

much easier

$$= \lim_{t\to\infty} \int_{10}^{t} \frac{1}{x^5}\,dx$$

$$= \lim_{t\to\infty} -\frac{1}{4x^4}\Big|_{10}^{t}$$

$$= 0 - \left(-\frac{1}{40{,}000}\right)$$

$$= \frac{1}{40{,}000} = .000025.$$

So: combining the above gives:

$$R_{10} \le \int_{10}^{\infty} \frac{1}{x^5+5}\,dx \le \int_{10}^{\infty} \frac{1}{x^5}\,dx$$

$$= .000025$$

So, our estimate $s_{10} = .19926\ldots$ is w/in $.000025$ of the actual series $\sum_{n=1}^{\infty} \frac{1}{n^5+5}$ ✓

①

Alternating Series

Def'n an alternating series is a series whose terms alternate in sign, i.e. a series of one of the following forms:

$$\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

or $$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

where $b_n \geq 0$.

ex's: $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

and $-2 + 4 - 8 + 16 - \cdots = \sum_{n=1}^{\infty} (-1)^n 2^n$

are alternating.

But: $1/2 + 1/4 - 1/8 + 1/16 + 1/32 - 1/64 + \cdots$

is not (needs to alternate every other term).

Q: When do alternating series converge?

② → answer is usually easier to determine than for a general series

Recall :- the convergence of a series $\sum_{n=1}^{\infty} a_n$ implies $\lim_{n\to\infty} a_n = 0$.

- converse not true: $\lim_{n\to\infty} a_n = 0$ does not imply $\sum_{n=1}^{\infty} a_n$ converges in general

(e.g $\sum \frac{1}{n}$).

but for alternating series, converse is true, ... as long as terms are decreasing in absolute value.

Theorem: Given an alternating series

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$ $(b_n > 0)$

IF: ① $b_n \geq b_{n+1}$ for all n

② $\lim_{n\to\infty} b_n = 0$

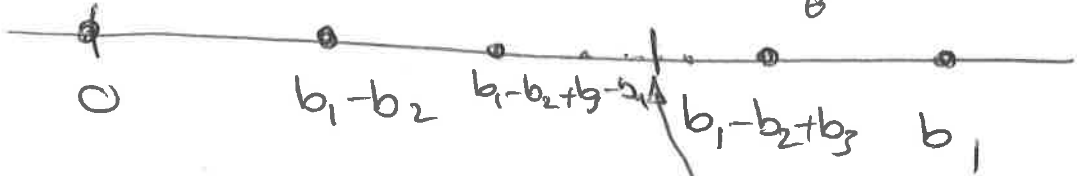
Then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges

(same true for $\sum (-1)^n b_n$ form)

③ "Proof by picture"

doesn't go below 0 since $b_2 \le b_1$

doesn't go above b_1 since $b_3 \le b_2$



we knew $b_2 \le b_1$

and $b_3 \le b_2$

$b_4 \le b_3$

$\vdots$

$L = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$

and each subsequent distance is getting smaller since $\lim_{n\to\infty} b_n = 0$

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ex's ① the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

converges:

why: ① $\frac{1}{n+1} \le \frac{1}{n}$ for all n

② $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

↳ Thm says: series converges.

(a) (u)

② the series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{n+1} = \frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \cdots$

is alternating.

But does not converge:

$\lim_{n\to\infty} (-1)^{n-1}\frac{n}{n+1}$ does not $= 0$

So theorem doesn't apply.

In fact, since $\lim_{n\to\infty} \frac{n}{n+1} = 1$

we have $\lim_{n\to\infty} (-1)^{n-1}\frac{n}{n+1}$ DNE.

So by the divergence test

$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n}{n+1}$ diverges.

③ What about $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$?

Sol'n: It "looks" like ne^{-n} is decreasing, but is it? Yes.

observe: $\frac{d}{dx} x e^{-x} = \frac{d}{dx}\frac{x}{e^x} = \frac{e^x - xe^x}{e^{2x}}$

$= \frac{1-x}{e^x}$

(5) We have that $\frac{1-x}{e^x} \leq 0$ if $x \geq 1$

So x/e^x is decreasing on $[1,\infty)$

$\Rightarrow n/e^n = ne^{-n}$ is too, i.e.

(1) $n+1e^{-(n+1)} \leq ne^{-n}$ for all $n \geq 1$

furthermore $\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n}$

$= \lim_{x\to\infty} \frac{x}{e^x}$

$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$. (2)

So (1) +(2) $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} ne^{-n}$ converges, by thm