①

Approximating solutions numerically

<u>Situation</u>: have diff'eq

$$y' = F(x,y)$$

we can't solve (not separable/etc) for y.

<u>Want</u>: to approximate a particular sol'n $y = f(x)$ thru a given point (x_0, y_0) (i.e. satisfying $y(x_0) = y_0$) numerically.

<u>idea</u>: use eq'n (gives info about f') to approximate f.

<u>ex</u>: consider eq'n

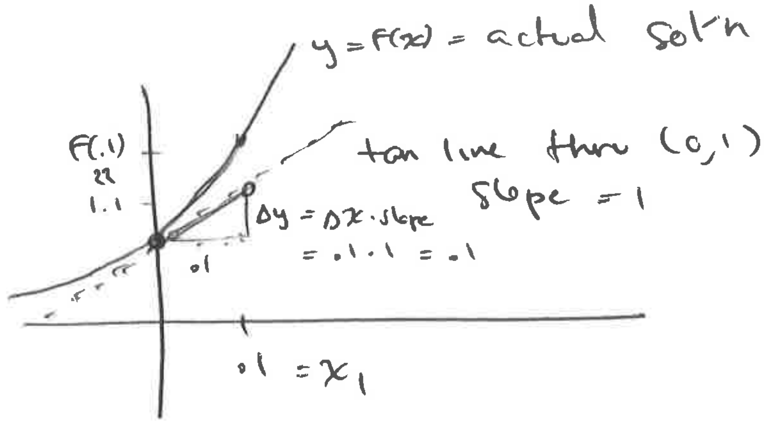
$$y' = x + y.$$

Sps we want to approx. sol'n $y = f(x)$ thru $(0,1)$ (i.e. satisfying $y(0) = 1$)

$\underbrace{(0,1)}_{= (x_0, y_0)}$

eq'n tells us: tan line to $y = f(x)$ there has slope $y' = x + y = 0 + 1 = 1$.

② – tan line gives rough approx'n to f near (0,1)



if we want to approx f(.1), we can follow tan line

$$f(.1) \approx y_0 + \Delta y$$

$\leftarrow \because y' = x_0 + y_0 = 1$

$$= 1 + (\text{slope})\Delta x$$

$$= 1 + 1(.1) = 1.1$$ ← call this y_1

– what if we want to approximate f(.2)?

– could use orig. tan line, but as we get further from $x_0 = 0$ this tan line becomes a worse approx'n for f.

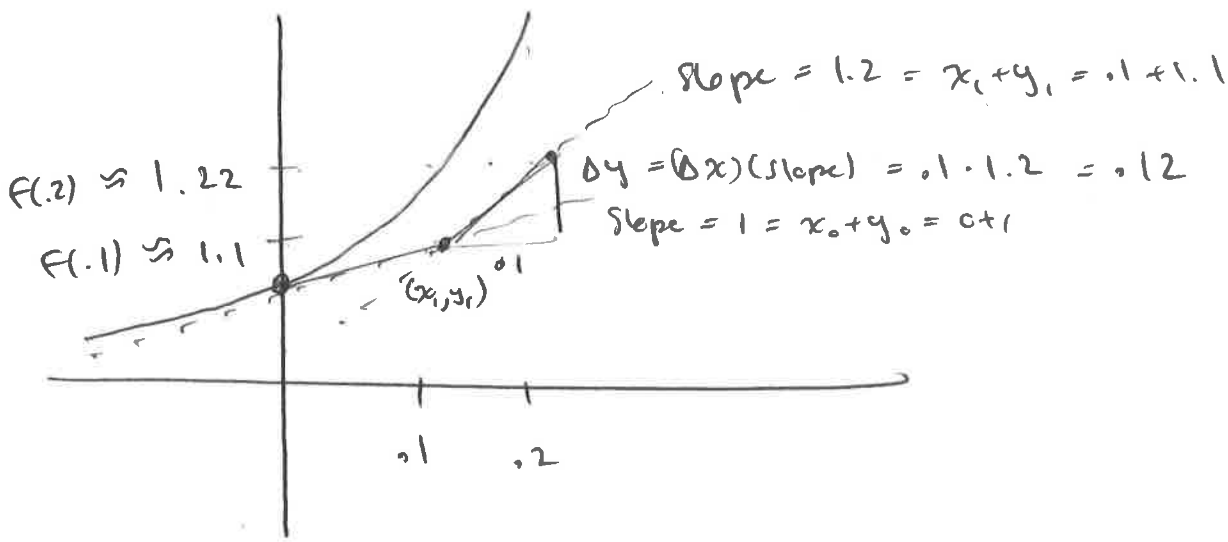
– instead: let's use diff'eq to get a new slope @ $(x_1, y_1) = (.1, 1.1)$

③

- there we get

$$y' = x + y = x_1 + y_1 = .1 + 1.1$$
$$= 1.2$$

- to approx. $f(.2)$ we follow the tan line w/ this slope beginning @ $(x_1, y_1) = (.1, 1.1)$



we get $f(.2) \approx y_1 + \Delta y$

$y_1' = x_1 + y_1 = 1.2$

$$= 1.1 + (\text{slope})\Delta x$$
$$= 1.1 + (1.2)(.1)$$
$$= 1.22$$

④ can repeat to approx $f(.3)$

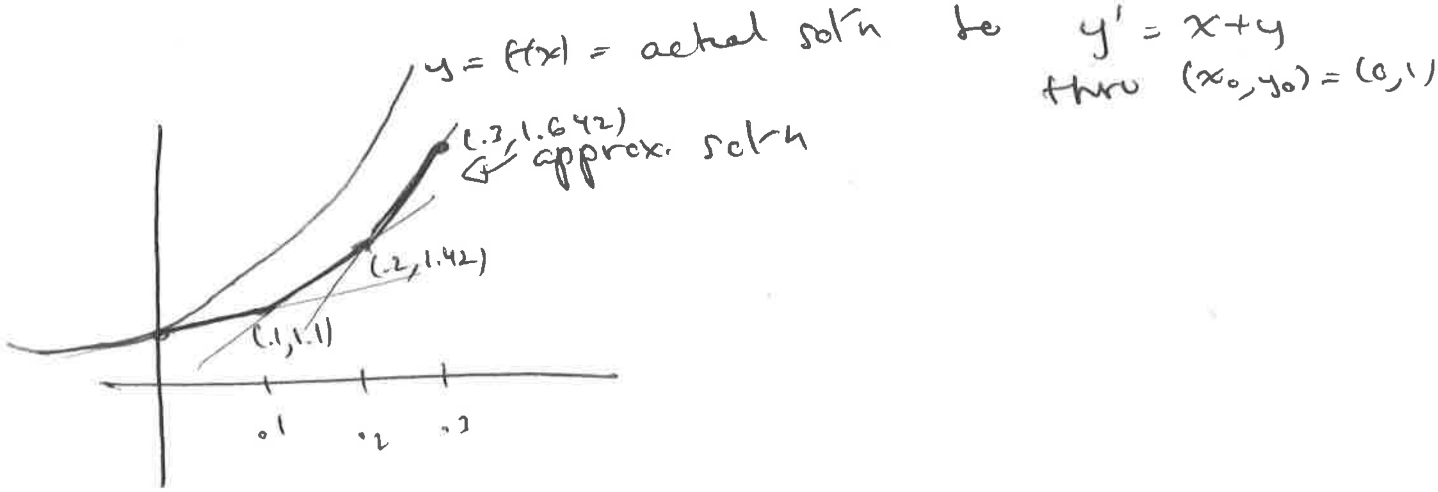
$$f(.3) \approx y_2 + \Delta y$$

$y' = x_2 + y_2$
$= .2 + 1.22 = 1.42$

$$= 1.22 + (\text{slope})\Delta x$$

$$= 1.22 + (1.42)(.1)$$

$$= 1.642$$



In general, to get approximate values $y_n \approx f(x_n)$ for the solution $y = f(x)$ to eq'n $y' = F(x,y)$ thru (x_0, y_0) we have:

"step size"

$$y_n = y_{n-1} + \Delta x\, F(x_{n-1}, y_{n-1})$$

new y old y slope from (x_{n-1}, y_{n-1})

(5)

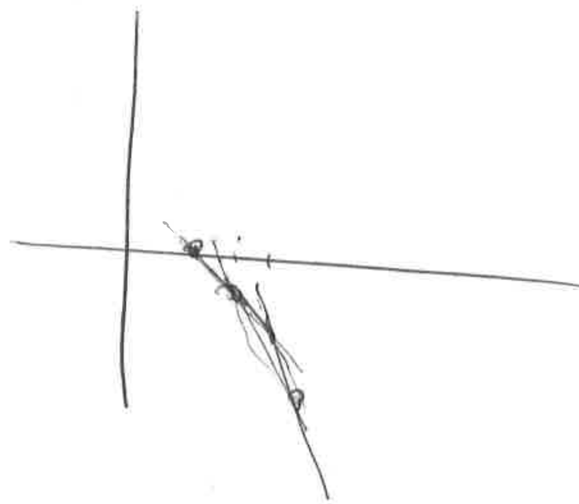
ex: (9.3.21) Using step size .5, approximate y values y_1, y_2, y_3 for sol'n of eq'n $y' = y - 2x$ satisfying $y(1) = 0$.

Sol'n: - Know: $y(1) = 0$ $\quad y' = y - 2x$

- Want to approx: $y(1.5), y(2), y(2.5)$

$$\begin{array}{ccc} y(1.5) & y(2) & y(2.5) \\ \approx & \approx & \approx \\ y_1 & y_2 & y_3 \end{array}$$

$$y(1.5) \approx y_1 = y_0 + \overset{\Delta x}{.5}\,(y'(1,0))$$
$$= 0 + .5\,(0 - 2 \cdot 1)$$
$$= 0 - 1 = -1$$



$$y(2) \approx y_2 = y_1 + .5\,(y'(1.5,-1))$$
$$= -1 + .5\,(-1 - 2(1.5))$$
$$= -3$$

$$y(2.5) \approx y_3 = y_2 + .5\,(y'(2,-3))$$
$$= -3 + .5\,(-3 - 2(2))$$
$$= -3 + -3.5 = -6.5$$

11.1 Sequences

— a <u>sequence</u> is an infinite list of numbers, indexed by positive integers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

<u>ex</u>: 1) $\overset{a_1}{2}, \overset{a_2}{4}, \overset{a_3}{6}, 8, 10, \ldots$

2) $1, 1/2, 1/3, 1/4, \ldots$

3) $1, -1, 1, -1, \ldots$

4) $1, 11, 21, 1211, 111221, 312211, \ldots$

are sequences.

— Sometimes the nth term a_n of a sequence can be computed by a formula

e.g. for sequences above, we have

1) $a_n = 2n$
2) $a_n = 1/n$
3) $a_n = (-1)^{n+1}$
4) no nice fmla for a_n (what is pattern?)

②

- sequences, like functions, can have <u>limits</u>
- the notation

$$\lim_{n \to \infty} a_n = L$$

intuitively means:

"as n gets larger, a_n gets arbitrarily close to L"

E.g. consider the sequence

$$1/2, \ 2/3, \ 3/4, \ 4/5, \ \ldots \qquad \left(a_n = \frac{n}{n+1}\right)$$

Intuitively for this sequence we have

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n+1} = 1$$

To <u>prove</u> this, need a <u>rigorous</u> def'n of limit.

<u>Def'n</u> We say the sequence a_n has limit L, and write $\lim_{n \to \infty} a_n = L$, if:

(3)

For every $\varepsilon > 0$
there is an N
such that for every $n > N$
we have $|a_n - L| < \varepsilon$

"no matter how close I want to get to L"
"I can go out far enough in my sequence"
"such that at ANY later point"
"I'm at least that close to L"

Ex: Claim: Let $a_n = \frac{n}{n+1}$ be the sequence above. Then $\lim_{n \to \infty} a_n = 1$.

PF: - Fix $\varepsilon > 0$.
- Then we can find a fixed N s.t. $\frac{1}{N} < \varepsilon$.
- Then for any $n > N$, observe we have $\frac{1}{n} < \frac{1}{N}$.
- Hence, for any $n > N$, we have:

$$
\begin{aligned}
|a_n - 1| &= \left|\frac{n}{n+1} - 1\right| \\
&= \left|\frac{n}{n+1} - \frac{n+1}{n+1}\right| \\
&= \left|-\frac{1}{n+1}\right| \\
&= \frac{1}{n+1} < \frac{1}{n} < \frac{1}{N} < \varepsilon
\end{aligned}
$$

Since ε was arbitrary (i.e. we could have repeated same arg for any ε) the claim is proved.

Summary of the proof: we showed no matter how small ε is to begin with, we can find a point in our sequence a_N, such that all subsequent terms a_n are within ε of 1. Hence 1 is the limit.

Terminology: - if $\lim_{n \to \infty} a_n = L$ we say a_n converges to L.

- if a_n does not have a limit we say the sequence diverges.

→ can also define what it means for a_n to have a limit of ∞ or $-\infty$:

$\lim_{n \to \infty} a_n = \infty$ means: "as n gets larger, a_n becomes arbitrarily large"

(5) $\lim_{n\to\infty} a_n = -\infty$ means: "as n gets larger, a_n becomes arbitrarily negative".

↳ see book for formal def'n.

- In practice, we use theorems that allows us to compute limits w/o using ε definition

<u>Thm (limit laws)</u> Sps a_n and b_n are convergent series and c is a constant. Then:

- $\lim_{n\to\infty} c = c$
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} a_n \cdot b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} a_n \Big/ \lim_{n\to\infty} b_n$ as long as $\lim_{n\to\infty} b_n \neq 0$

⑥

- if $f(x)$ is a continuous function

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) \qquad (*)$$

- in particular: if p is fixed number:

$$\lim_{n\to\infty} (a_n)^p = (\lim_{n\to\infty} a_n)^p$$

(why: $f(x) = x^p$ is continuous)

<u>ex</u>: Claim: $\lim_{n\to\infty} \frac{n}{n+1} = 1$

Pf: already verified w/ def'n, but easier w/ limit laws:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+1}$$

$$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} 1}$$

$$= \frac{1}{0+1} = 1 \checkmark$$

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Here's another useful fact for computing limits:

<u>Thm</u>: if $f(x)$ is a function and we define a sequence by $a_n = f(n)$, then:

$$\text{if } \lim_{x\to\infty} f(x) = L$$

$$\text{then } \lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L \text{ also.}$$

↳ this theorem allows us to exploit L'hospital's rule to compute limits of certain sequences.

<u>ex</u>: <u>Claim</u>:

$$\lim_{n\to\infty} n^2 e^{-n^2} = 0$$

Pf.
$$\lim_{n\to\infty} n^2 e^{-n^2} = \lim_{n\to\infty} \frac{n^2}{e^{n^2}} \quad \text{“}\frac{\infty}{\infty}\text{”}$$

$$\overset{\text{by thm above}}{=} \lim_{x\to\infty} \frac{x^2}{e^{x^2}}$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2x}{2xe^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2}} = 0 \checkmark$$

(8) using (*) to compute limits

<u>ex</u>: Compute the limits of the sequences, or show they diverge:

1) $a_n = \cos(1/n)$

2) $a_n = \tan(\pi n / 2n+1)$

3) $a_n = \sin\left(\frac{\pi n}{2}\right)$

note: $\cos(x)$, $\sin(x)$, and $\tan(x)$ are cts on their domains

also note: $\lim_{n\to\infty} \frac{1}{n} = 0$ $\quad \lim_{n\to\infty} \frac{\pi n}{2n+1} = \frac{\pi}{2}$

$\lim_{n\to\infty} \pi n = \infty$

<u>sol'n</u>: by (*)

1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(1/n)$

$\overset{(*)}{=} \cos\left(\lim_{n\to\infty} 1/n\right)$

$= \cos(0) = 1$

2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \tan(\pi n/2n+1)$

$\underset{?}{\overset{(*)}{=}} \tan\left(\lim_{n\to\infty} \pi n/2n+1\right)$

$= \pi/2$, not in domain, so (*) doesn't

(9)

still can evaluate by inspection:

$$\lim_{n\to\infty} \tan\left(\pi n/2n+1\right)$$

approaches $\pi/2$

approaches ∞

$= \infty$

3) (*) doesn't apply since $\lim_{n\to\infty} \pi n = \infty$.

can evaluate by inspection.

First few terms are

$a_1 = \sin(\pi/2) = 1$

$a_2 = \sin(2\pi/2) = 0$

$a_3 = \sin(3\pi/2) = -1$

$a_4 = \sin(4\pi/2) = 0$

$= 1$

$= 0$

$\vdots$

So seq. oscillates $\Rightarrow$ limit DNE.

(c) <u>Squeeze theorem</u>:

if a_n, b_n, c_n are sequences s.t.

$a_n \le b_n \le c_n$ for every n,

then:

if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$ as well

<u>ex</u>: Find $\lim_{n\to\infty} \frac{\cos(e^n)}{n}$

<u>sol'n</u>: observe: $-1 \le \cos(e^n) \le 1$ for every n

hence $\underbrace{-\frac{1}{n}}_{a_n} \le \underbrace{\frac{\cos(e^n)}{n}}_{b_n} \le \underbrace{\frac{1}{n}}_{c_n}$ for every n

and since

$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

by squeeze theorem

$\lim_{n\to\infty} \cos(e^n)/n = 0$ as well.

(11) The sequence $a_n = r^n$

if $r = 2$: $a_n = 2^n$: $2, 4, 8, 16 \ldots$

$\lim_{n\to\infty} 2^n = \infty$ diverges

if $r = 1$ $a_n = 1^n = 1$: $1, 1, 1, \ldots$

$\lim_{n\to\infty} 1^n = 1$ converges

if $r = 1/2$ $a_n = \left(\frac{1}{2}\right)^n$: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

if $r = -1$ $a_n = (-1)^n$: $-1, 1, -1, \ldots$

$\lim_{n\to\infty} (-1)^n$ DNE.

in general: $\lim_{n\to\infty} r^n = 0$

- if $-1 < r \leq 1$

- diverges otherwise

(12) Monotone convergence

Def'n: a sequence is increasing if $a_{n+1} > a_n$ for every n

(e.g. $1/2, 2/3, 3/4, \ldots, n/n+1, \ldots$)

" " decreasing

if $a_{n+1} < a_n$ for every n

(e.g. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$)

" " monotone

if it is either increasing or decreasing

" " bounded

if there is a constant C s.t. $|a_n| \leq C$ for every n.

e.g. if $a_n = \left(-\frac{1}{2}\right)^n$: $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

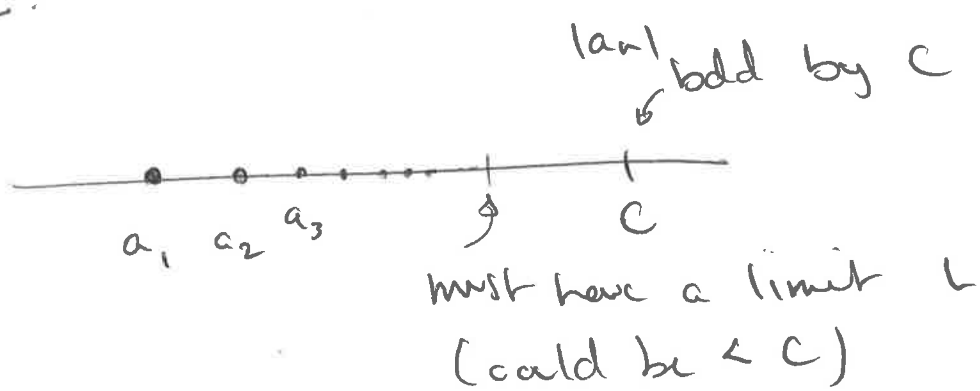
then $|a_n| \leq 1$

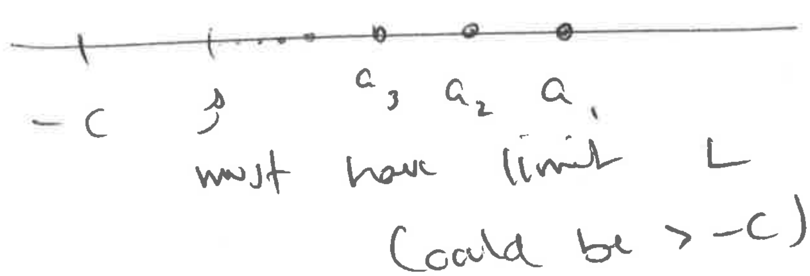
Say: sequence is bounded by 1.

(13) <u>Theorem</u> (monotone convergence)

If a_n is monotone and bounded then the sequence converges (i.e. $\lim_{n\to\infty} a_n$ exists).

<u>Picture</u>:





↳ theorem is useful for allowing us to apply limit laws to <u>recursively defined</u> sequences

(14)

Recursively defined sequences

ex. Define a sequence by:

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

so e.g.

$$a_2 = 3 - \frac{1}{1} = 2$$
$$a_3 = 3 - \frac{1}{2} = 5/2 = 2.5$$
$$a_4 = 3 - \frac{2}{5} = 13/5 = 2.6$$
$$a_5 = 3 - 5/13 = 34/13 = 2.615$$
$$\vdots$$

- It can be shown (by induction) that a_n is increasing.
- Further $|a_n| \leq 3$ so by monotone convergence, a_n has a limit.

But what is it?

observe: $\lim_{n\to\infty} a_n \overset{\text{why?}}{=} \lim_{n\to\infty} a_{n+1}$

$$\overset{\text{def'n}}{=} \lim_{n\to\infty} 3 - \frac{1}{a_n}$$

$$\overset{\text{limit laws}}{=} 3 - \frac{1}{\lim_{n\to\infty} a_n}$$

(15) Let $L = \lim_{n \to \infty} a_n$

we've just shown

$$L = 3 - \frac{1}{L}$$

so: $L^2 = 3L - 1$

$\Rightarrow L^2 - 3L + 1 = 0$

$\Rightarrow L = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3 \pm \sqrt{5}}{2}$

So which is our actual limit

$\dfrac{3+\sqrt{5}}{2}$ or $\dfrac{3-\sqrt{5}}{2}$

$= 2.618\ldots$

must be this since our seq. is increasing and hence $L > a_1 = 2 > \dfrac{3-\sqrt{5}}{2}$

11.2 Series

Given a sequence a_n, the associated series is the "infinite sum"

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

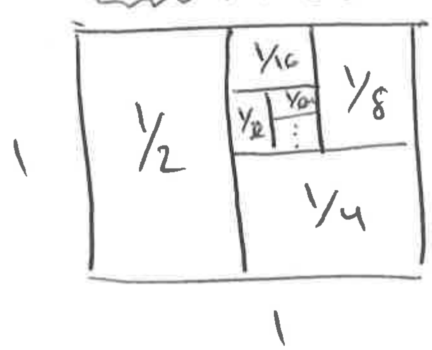
↳ does this concept make sense?
↳ isn't such a sum always $= \infty$?

ex: Consider the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$= \infty$? not quite:

THE UNIT SQUARE! Area $= 1 \times 1 = 1$



Seems like $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ = area of square

$= 1$.

② To make precise, need to define convergence of a series.

Def'n Given a series $\sum_{n=1}^{\infty} a_n$, the sequence of partial sums S_n is defined by:

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

- if $\lim_{n\to\infty} S_n = L$ we say the series $\sum_{n=1}^{\infty} a_n$ converges to L and write $\sum_{n=1}^{\infty} a_n = L$

- if $\lim_{n\to\infty} S_n$ does not exist (or $= \pm\infty$) we say the series $\sum_{n=1}^{\infty} a_n$ diverges

ex: for the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

the first few partial sums are:

$$S_1 = a_1 = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

③

$S_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$\vdots$

$S_n = \frac{2^n - 1}{2^n}$

Can compute: $\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} \frac{2^n - 1}{2^n}$

$= \lim\limits_{n \to \infty} 1 - \frac{1}{2^n}$

$= 1$

Hence our series converges to 1.

We write: $\sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1$.

__ex__: the series

$1 + 1 + 1 + \cdots = \sum\limits_{n=1}^{\infty} 1$

does not converge since in this case

$S_n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n$

So $\lim\limits_{n \to \infty} S_n = \lim\limits_{n \to \infty} n = \infty$.

(4)

Def'n: a geometric series is a series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

where a, r are real numbers

e.g.

$$\sum_{n=1}^{\infty} 5(1/3)^{n-1} = 5 + 5/3 + 5/9 + 5/27 + \cdots$$

$$\sum_{n=1}^{\infty} \tfrac{1}{2}(\tfrac{1}{2})^{n-1} = \sum_{n=1}^{\infty} (\tfrac{1}{2})^n = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

$$\sum_{n=1}^{\infty} 3 \cdot 2^{n-1} = 3 + 6 + 12 + 24 + \cdots$$

are geometric series.

Thm The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1}$$

converges if and only if $|r| < 1$.
In this case we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

(5) Pf: assume $|r|<1$. (If $|r|\geq 1$, clear sum diverges).

nth partial sum given by:

$$S_n = \sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \dots + ar^{n-1}$$

$$= a(1 + r + r^2 + \dots + r^{n-1})$$

$$\Rightarrow (1-r)S_n = a(1 + r + \dots + r^{n-1})(1-r)$$

$$= a(1 + \cancel{r} + \cancel{r^2} + \dots + \cancel{r^{n-1}}$$
$$- \cancel{r} - \cancel{r^2} - \cancel{r^3} - \dots - \cancel{r^{n-1}} - r^n)$$

$$= a(1-r^n)$$

$$\Rightarrow S_n = \frac{a(1-r^n)}{1-r}$$

$$\text{hence } \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-\overbrace{r^n}^{\to 0 \text{ since } |r|<1})}{1-r}$$

$$= \frac{a}{1-r}$$

By def'n: $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$

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<u>ex</u>: 1) $\sum_{n=1}^{\infty} 5(1/3)^{n-1} = 5 + 5/3 + 5/9 + \cdots$

$\overset{\text{thm}}{=} \dfrac{5}{1-1/3} = 15/2$

2) $\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n}$

$= \dfrac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

3) $\sum_{n=1}^{\infty} 3(-1/2)^{n-1} = 3 - 3/2 + 3/4 - 3/8 + \cdots$

$= \dfrac{3}{1-(-1/2)} = \dfrac{3}{\frac{3}{2}} = 2$

4) $\sum_{n=1}^{\infty} 3 \cdot 2^{n}$ = diverges $(2 > 1)$

<u>ex</u>: every real number with an eventually repeating decimal expansion can be written as a fraction!

e.g. consider

$1.2343434\ldots = 1.2\overline{34}$

$= 1.2 + .034 + .00034 + .0000034 + \cdots$

$= 1.2 + \dfrac{34}{1000} + \dfrac{34}{100000} + \dfrac{34}{10000000} + \cdots$

(7)

$$= 1.2 + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \cdots$$

$$= 1.2 + \frac{34}{10^3} + \frac{34}{10^3}\left(\frac{1}{10^2}\right) + \frac{34}{10^3}\left(\frac{1}{10^2}\right)^2 + \frac{34}{10^3}\left(\frac{1}{10^2}\right)^3 + \cdots$$

$$= 1.2 + \sum_{n=1}^{\infty} \frac{34}{10^3}\left(\frac{1}{10^2}\right)^{n-1}$$

$$= 1.2 + \frac{\frac{34}{10^3}}{\underbrace{1 - \frac{1}{10^2}}_{= \frac{99}{100}}} = 1.2 + \frac{34}{1000} \cdot \frac{100}{99}$$

$$= 1.2 + \frac{34}{990}$$

$$= \frac{12}{10} + \frac{34}{990}$$

$$= \frac{1188}{990} + \frac{34}{990} = \frac{1222}{990}$$

$$= 1.23434\ldots \checkmark$$

<u>ex</u>: (telescoping series)

- not all series are geometric of course
- sometimes can show convergence w/ a clever trick.

Consider:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$$

$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

⑧ - not geometric - does it converge?

<u>Observe</u>: $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$

so our nth partial sum is:

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right)$$

$$\cancel{+} \cdot \cancel{+} \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

hence

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

so we've shown:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \checkmark$$

<u>ex</u> (the harmonic series diverges)

Consider $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

- not geometric: does it converge?

(9) Consider the partial sum

$S_2, S_4, S_8, S_{16}, \ldots, S_{2^n}, \ldots$

$S_2 = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$

$= 1 + \frac{1}{2} + \frac{1}{2} = 2$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2.5$

$S_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots + \frac{1}{16}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \geq 3$

in general: $S_{2^n} > 1 + \frac{n}{2}$

hence $\lim_{n \to \infty} S_n = \infty$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges!

(10) Thm (divergence test)

If $\sum_{n=1}^{\infty} a_n$ converges,

then $\lim_{n\to\infty} a_n = 0$.

Pf: "clear" (see book)

<u>the point</u>: <u>if</u> $\lim_{n\to\infty} a_n \neq 0$ (i.e. converges to another limit or DNE)

<u>then</u> $\sum_{n=1}^{\infty} a_n$ <u>must</u> diverge!

<u>ex</u>: the sum

$$\sum_{n=1}^{\infty} \frac{n^3+1}{n(n^2+1)}$$

diverges:

<u>why</u>:
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3+1}{n(n^2+1)}$$
$$= \lim_{n\to\infty} \frac{n^3+1}{n^3+n}$$
$$= \lim_{n\to\infty} \frac{1+\frac{1}{n^3}}{1+\frac{1}{n^2}}$$
$$= 1 \neq 0$$

(11) Summation rules

Thm: if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge then

$$\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \checkmark$$

$$\sum_{n=1}^{\infty} c a_n = c\sum_{n=1}^{\infty} a_n \checkmark$$

ex) $\sum_{n=1}^{\infty}\left(5\left(\frac{1}{2}\right)^{n-1} - 3\left(\frac{1}{7}\right)^{n-1}\right)$

$$= \sum_{n=1}^{\infty} 5\left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} 3\left(\frac{1}{7}\right)^{n-1}$$

$$= \frac{5}{1-\frac{1}{2}} - \frac{3}{1-\frac{1}{7}}$$

$$= 10 - \frac{7}{2}$$

$$= \frac{13}{2} \checkmark$$