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#33 $\int \sqrt{3-2x-x^2}\, dx$

$= \int \sqrt{-(x^2+2x-3)}\, dx$

$= \int \sqrt{-[(x+1)^2-4]}\, dx$

$= \int \sqrt{4-(x+1)^2}\, dx$

$x+1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$= \int \sqrt{(2\cos\theta)^2}\, 2\cos\theta\, d\theta$

$= \int 4\cos^2\theta\, d\theta$

$= 2\int 1+\cos 2\theta\, d\theta$

$= 2\left(\theta + \frac{1}{2}\sin 2\theta\right)$

$= 2\left(\theta + \sin\theta\cos\theta\right)$

$= 2\left(\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{(x+1)\sqrt{4-(x+1)^2}}{4}\right) + C$

#81.

$$\int \sqrt{1-\sin x}\,dx$$

$$= \int \sqrt{\frac{(1-\sin x)(1+\sin x)}{1+\sin x}}\,dx$$

$$= \int \sqrt{\frac{1-\sin^2 x}{1+\sin x}}\,dx$$

$$= \int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$$

$u = 1+\sin x$

$du = \cos x\,dx$

$$= \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}$$

$$= 2u^{1/2}$$

$$= 2(1+\sin x)^{1/2}$$

$$= \boxed{2\sqrt{1+\sin x} + C}$$

7.7. Approximating Integrals

- many integrable functions $f(x)$ for which $\int f(x)\,dx$ can't be determined (in terms of elem. functions)

- Can still get good estimates of $\int_a^b f(x)\,dx$ using Riemann sums.

- Come in various flavors:
  - left endpoint approximation
  - right e.p. approx.
  - midpoint rule
  - trapezoidal rule.

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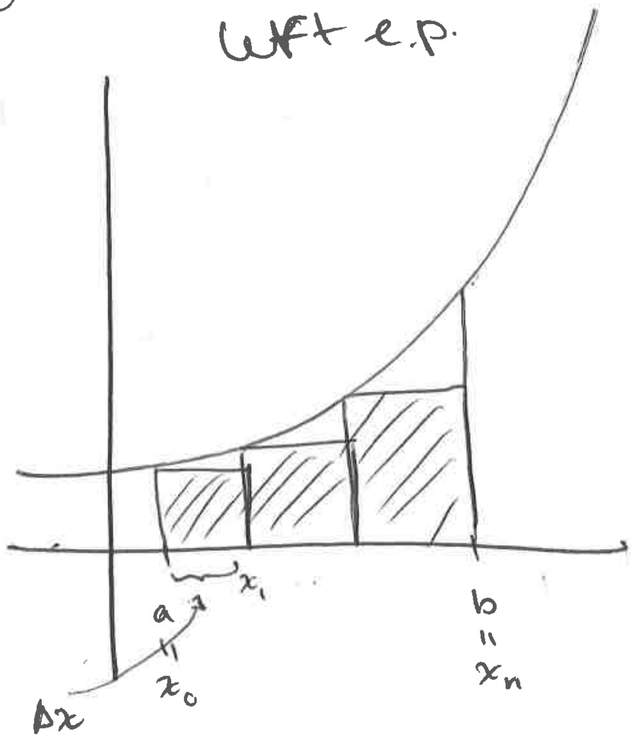
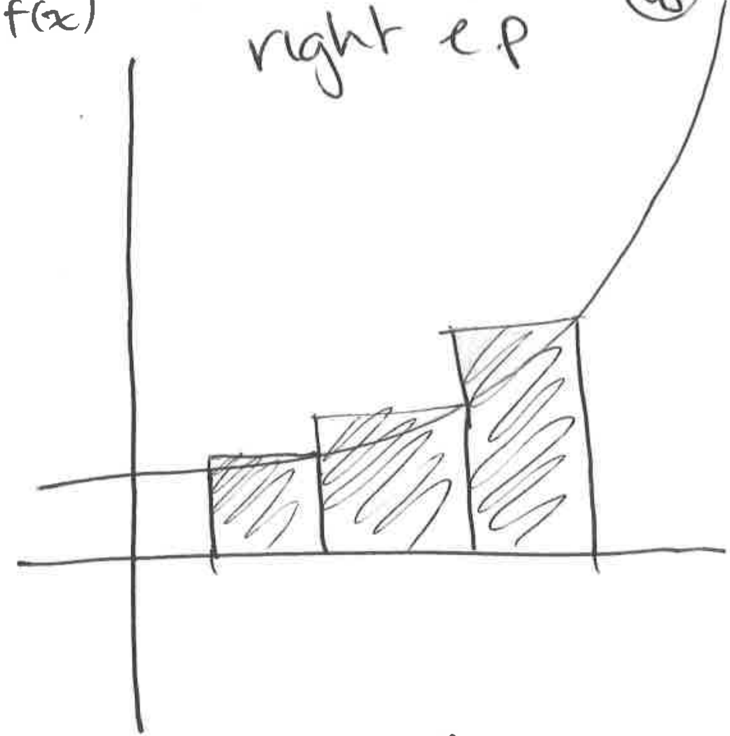
left e.p.



right e.p



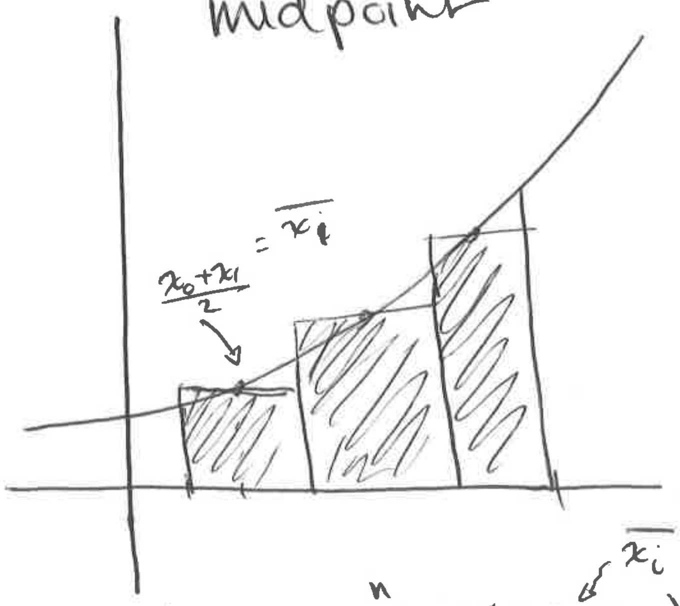
$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$= \Delta x\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right)$$

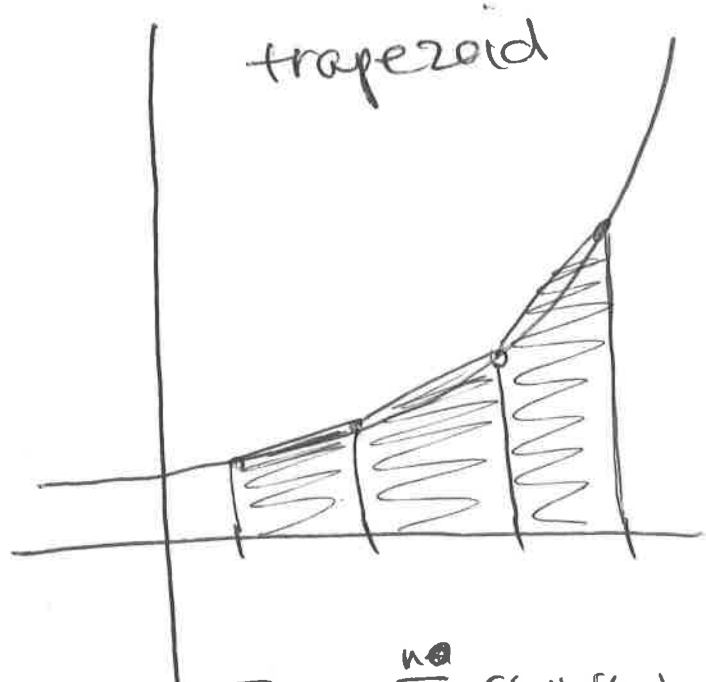
$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \Delta x\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)$$

midpoint



trapezoid



$$M_n = \sum_{i=1}^{n} f\Bigl(\overbrace{\frac{x_i + x_{i-1}}{2}}^{\bar{x_i}}\Bigr)\Delta x$$

$$= \sum_{i=1}^{n} f(\bar{x_i})\Delta x$$

$$T_n = \sum_{i=1}^{n} \frac{f(x_i) + f(x_{i-1})}{2}\Delta x$$

$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

- all of these methods yield the integral in the limit:

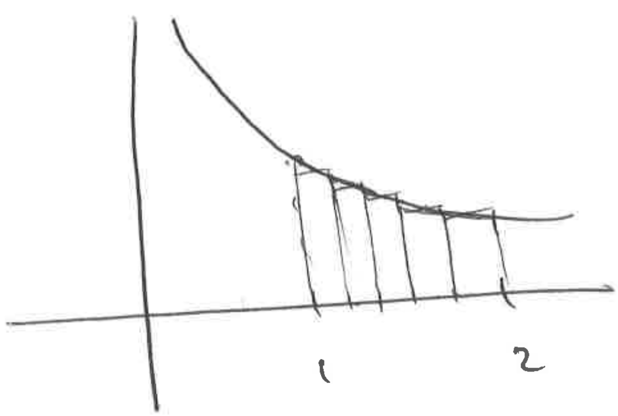
$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n = \lim_{n\to\infty} T_n$$

$$= \int_a^b f(x)\,dx$$

- but for a given n, some better approximate $\int_a^b f(x)\,dx$ than others.

ex use trapezoidal rule and midpoint rule to approx. area under $\frac{1}{x}$ between $x=1,2$

Use $n=5$.



$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = 1/5 = .2$$

$$x_0 = 1,\ 1.2,\ 1.4,\ 1.6,\ 1.8,\ 2 = x_5$$

$$\quad x_1 \quad x_2 \quad x_3 \quad x_4$$

(a) midpoints: $\bar{x}_1$ 1.1, $\bar{x}_2$ 1.3, $\bar{x}_3$ 1.5, $\bar{x}_4$ 1.7, $\bar{x}_5$ 1.9

Trap rule gives:

$$\int_1^2 \frac{1}{x}\,dx \approx T_5 = \frac{\Delta x}{2}\Big(f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\Big)$$

$$= \frac{.2}{2}\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right)$$

$$= .6956\ldots$$

Midpoint gives:

$$\int_1^2 \frac{1}{x}\,dx = M_5 = \Delta x\big(f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\big)$$

$$= .2\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$= .6919\ldots$$

if we actually compute

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_{x=1}^{2} = \ln(2) - \ln(1) = \ln(2)$$

$$= .6931\ldots$$

↳ trap rule overshoots, m.p. rule undershoots, m.p. a bit closer overall.

↳ in general: trap and m.p. both better than l.e.p and r.e.p.

↳ m.p. tends to be better than trap.

to make this precise we define the errors:

$$E_T = \int_a^b f(x)\,dx - T_n$$

$$E_M = \int_a^b f(x)\,dx - M_n$$

$$E_L = \int_a^b f(x)\,dx - L_n$$

$$E_R = \int_a^b f(x)\,dx - R_n$$

so far ex we have:

$$E_T = \int_1^2 \frac{1}{x} - T_5 = \ln 2 - .6956\ldots = .0025\ldots$$

$$E_M = \int_1^2 \frac{1}{x} - M_5 = .0012\ldots$$

↳ E_M tends to be smaller (in abs value) than E_T, in following sense:

6 <u>Theorem</u> Suppose $|f''(x)| \le K$ on the interval $a \le x \le b$. Then:

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Note: these are <u>upper bound</u> estimates on the errors — actual errors may be smaller.

e.g. for estimating $\int_1^2 \frac{1}{x}\,dx$ with $n=5$ these give:

$$\frac{d^2}{dx^2}\frac{1}{x} = \frac{d}{dx}\frac{-1}{x^2} = \frac{2}{x^3}$$

observe $\frac{2}{2^3} \le \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2$ on $[1,2]$

hence estimate theorem gives:

$$E_T \le \frac{2(2-1)^3}{12\cdot 5^2} = .0066\ldots$$

$$E_M \le \frac{2(2-1)^3}{24\cdot 5^2} = .0033\ldots$$

but actual errors were

$$E_T = .0025\ldots$$

$$E_M = .0012\ldots$$

Still, we can use these errors estimates to answer questions like the following:

Q: How large must n be to guarantee that $|E_T|$ and $|E_M|$ are $\leq .0001$ for $\int_1^2 \frac{1}{x}\,dx$

Sol'n: For $|E_T|$ we know

$$|E_T| \leq \frac{2(1)^3}{12n^2} \leq .0001$$

$$\Rightarrow \frac{1}{6} \leq .0001n^2 = \frac{n^2}{10{,}000}$$

$$\Rightarrow \frac{10{,}000}{6} \leq n^2$$

$$\Rightarrow n \geq 40.8$$

So n must be at least 41 to guarantee the desired error for trap. rule.

For $|E_M| \leq \frac{2(1)^3}{24n^2} \leq .0001$

$$\Rightarrow n^2 \geq \frac{10{,}000}{12}$$

$$\Rightarrow n \geq 28.86$$ So $n \geq 29$ works.

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Another approx. rule – Simpson's rule – is more accurate than both trap. and m.p.

↳ for this rule we assume n is even and define

$$S_n = \frac{\overset{(b-a)/n}{\Delta x}}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

We define $E_S = \int_a^b f(x)\,dx - S_n$

Theorem: if $|f^{(4)}(x)| \leq K$ on the interval $a \leq x \leq b$, then

$$|E_S| \leq \frac{K(b-a)^5}{180\,n^4}$$

ex: Let's estimate $\int_1^2 (1/x)\,dx$ using Simpson with $n=10$. So $\Delta x = \frac{2-1}{10} = \frac{1}{10}$

$$S_{10} = \frac{1/10}{3}\big(f(1.1) + 2f(1.2) + 4f(1.3)$$
$$+ 2f(1.4) + 4f(1.5) + 2f(1.6)$$
$$+ 4f(1.7) + 2f(1.8) + 4f(1.9)$$
$$+ f(2.0)\big)$$

$$= .6931\ldots$$

in this case we have

$$E_S = \int_1^2 1/x\,dx - .6931\ldots$$

$$= \ln 2 - .6931\ldots$$

$$= -.00000282\ldots$$

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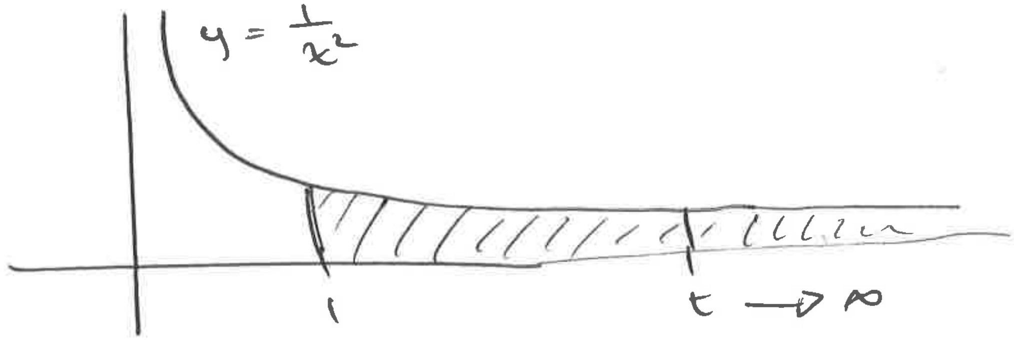
7.8 Improper Integration

Q: What is area under $\frac{1}{x^2}$ between 1 and ∞?



Is area infinite? Not so fast.

Observe: for any fixed $t \geq 1$ we have

$$\int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t$$

$$= -\frac{1}{t} + 1$$

So $\int_1^t \frac{1}{x^2}\,dx < 1$ for every t! and moreover

$$\underbrace{\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx}_{} = \lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

Can view this as area under $\frac{1}{x^2}$ from 1 to ∞ and denote $\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx := \int_1^\infty \frac{1}{x^2}\,dx = 1$

More generally, we define:

<u>Def'n</u>: Sps $\int_a^t f(x)\,dx$ exists for every $t \geq a$. Then define:

$$\int_a^\infty f(x) := \lim_{t\to\infty} \int_a^t f(x)\,dx$$

If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, define

$$\int_{-\infty}^b f(x) := \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

If there lims exist we say the integrals $\int_a^\infty f(x)\,dx$, $\int_{\infty}^b f(x)\,dx$ <u>converge</u>, if not, they <u>diverge</u>

Define

any fixed #

$$\int_{-\infty}^\infty f(x) := \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

This integral <u>converges</u> only if both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ converge.

③ Compute: (55)

1) $\int_1^\infty \frac{\ln(x)}{x}\,dx$

2) $\int_1^\infty \frac{\ln(x)}{x^2}\,dx$

3) $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

4) $\int_0^\infty \cos x\,dx$

Sol'n: 1) $\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x}\,dx$

$u = \ln x \quad du = \frac{1}{x}\,dx$

$\int u\,du$

$= \frac{1}{2}u^2$

$= \frac{1}{2}[\ln(x)]^2$

Recall:

$\lim_{t\to\infty} \ln(t) = \infty$

$= \lim_{t\to\infty} \left(\frac{1}{2} [\ln(x)]^2 \Big|_0^t \right)$

$= \lim_{t\to\infty} \left(\frac{1}{2} [\ln(t)]^2 - \frac{1}{2}[0] \right)$

$= \lim_{t\to\infty} \frac{1}{2} [\ln(t)]^2$

$= \infty$ diverges

(4) $\int_1^{\infty} \frac{\ln(x)}{x^2} = \lim_{t \to \infty} \int_1^t \frac{\ln(x)}{x^2}\,dx$ 56

$u = \ln x \quad dv = \frac{1}{x^2}$

$du = \frac{1}{x} \quad v = -\frac{1}{x}$

$$-\frac{1}{x}\ln(x) + \int \frac{1}{x^2}\,dx$$

$$= -\frac{1}{x}\ln(x) - \frac{1}{x} \Big|_1^t$$

$$= -\frac{1}{t}\ln t - \frac{1}{t} - (0 - 1) =$$

$$= 1 - \frac{\ln t}{t} - \frac{1}{t}$$

so overall

$$= \lim_{t \to \infty} \left(1 - \frac{\ln t}{t} - \frac{1}{t}\right)$$

$$= \lim_{t \to \infty} 1 - \lim_{t \to \infty} \frac{\ln t}{t} - \lim_{t \to \infty} \cancel{\frac{1}{t}}$$

$$\lim_{t \to \infty} 1 = 1 \qquad \lim_{t \to \infty} \frac{1}{t} = 0$$

L'hopital

$$= \lim_{t \to \infty} \frac{\frac{1}{t}}{1}$$

$$= 0$$

$$= 1 - 0 - 0 = 1 \checkmark$$

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(3) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

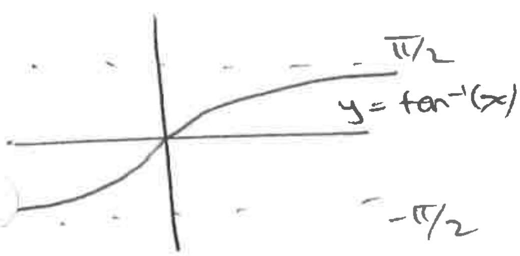
$$= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2} + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \left(\tan^{-1}(x)\Big|_t^0 \right) + \lim_{t\to\infty} \left(\tan^{-1}(x)\Big|_0^t \right)$$

$$= \lim_{t\to-\infty} \left(\underbrace{\tan^{-1}(0)}_{0} - \underbrace{\tan^{-1}(t)}_{-\pi/2} \right) + \lim_{t\to\infty} \left(\underbrace{\tan^{-1}(t)}_{\to\pi/2} - \underbrace{\tan^{-1}(0)}_{0} \right)$$

$$= -(-\pi/2) + \pi/2 = \pi \checkmark$$

Recall:



(4) $$\int_0^\infty \cos x\,dx = \lim_{t\to\infty} \int_0^t \cos x\,dx$$

$$= \lim_{t\to\infty} \left(\sin x \Big|_0^t \right)$$

$$= \lim_{t\to\infty} \left(\sin t - \underbrace{\sin 0}_{0} \right)$$

$$= \lim_{t\to\infty} (\sin t)$$

DNE!

$\Rightarrow$ integral diverges.

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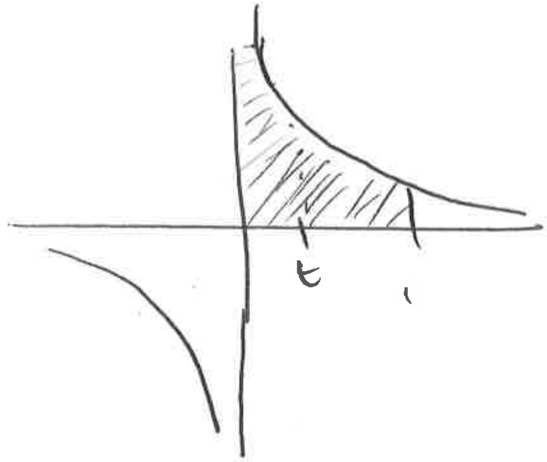
(6) Q: What is area under $\frac{1}{\sqrt{x}}$ between 0 and 1?



Observe: for any t with $0 < t \le 1$ we have:

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = \int_t^1 x^{-1/2}\,dx$$
$$= 2x^{1/2}\Big|_t^1$$
$$= 2\cdot 1 - 2\sqrt{t} = 2 - 2\sqrt{t}$$

So $\int_t^1 \frac{1}{\sqrt{x}} < 2$ for every t, $0 < t \le 1$

and moreover $\underbrace{\lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}}_{} = \lim_{t\to 0^+} 2 - 2\sqrt{t} = 2$

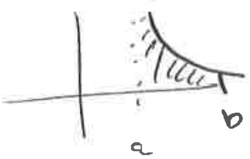
can interpret as area

denote: $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

looks honest... but is a limit!

More generally,

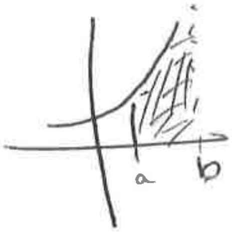
<u>Def'n</u> if f is continuous on $(a,b]$ w/ a discontinuity at a, define



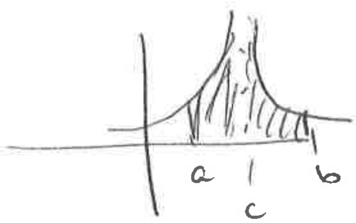
$$\int_a^b f(x)\,dx := \lim_{t\to a^+} \int_t^b f(x)\,dx$$

Similarly, if f cts on $[a,b)$ w/ a discontinuity at b, define

$$\int_a^b f(x) := \lim_{t\to b^-} \int_a^t f(x)\,dx$$



if f has a discontinuity at c with $a<c<b$ define



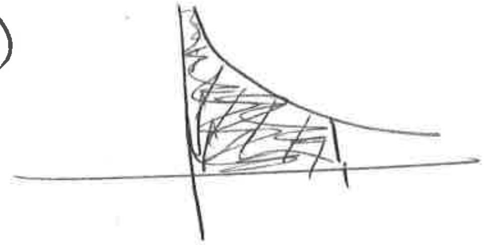
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

<u>ex</u>: Compute

1) $\int_0^1 \frac{1}{x^2}\,dx$

2) $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$

(8) Sol'n: 1)



$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$

$$= \lim_{t\to 0^+}\left[-\frac{1}{x}\Big|_t^1 \right]$$

$$= \lim_{t\to 0^+}\left[-1 + \frac{1}{t} \right]$$

$$= \lim_{t\to 0^+}(-1) + \underbrace{\lim_{t\to 0^+}\frac{1}{t}}_{\to\infty}$$

$$= \infty$$ diverge

2) first let's compute

$$\int \frac{1}{\sqrt[3]{x-1}}\,dx = \int (x-1)^{-1/3}\,dx$$

$$= \frac{3}{2}(x-1)^{2/3}$$

now $\frac{1}{\sqrt[3]{x-1}}$ has discontinuity @ $x=1$ which is between 0, 9.

So: $$\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

(9) $$= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{t\to 1^+} \int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

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$$= \lim_{t\to 1^-} \left(\frac{3}{2}(x-1)^{2/3} \Big|_0^t \right) + \lim_{t\to 1^+} \left(\frac{3}{2}(x-1)^{2/3} \Big|_t^9 \right)$$

$$= \lim_{t\to 1^-} \left(\underbrace{\frac{3}{2}(t-1)^{2/3}}_{0} - \frac{3}{2} \right) + \lim_{t\to 1^+} \left(\frac{3}{2}\cdot 4 - \underbrace{\frac{3}{2}(t-1)^{2/3}}_{0} \right)$$

$$= -\frac{3}{2} + \frac{3}{2}\cdot 4 = 6 - \frac{3}{2} = \frac{9}{2}$$