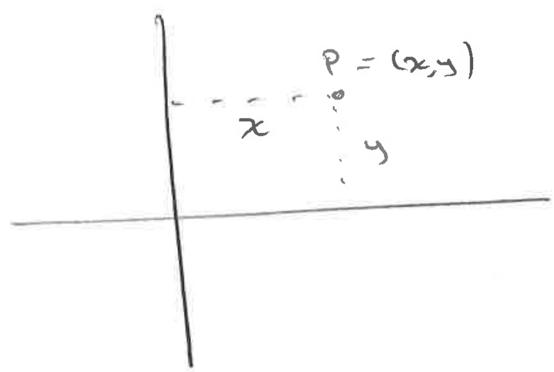
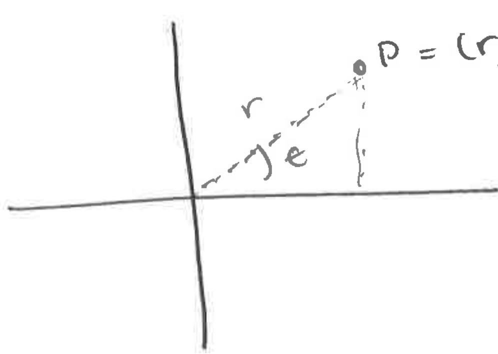
(i)

10.3 Polar Coordinates

- a point P in the plane is uniquely specified by its rectangular coordinates (x,y)

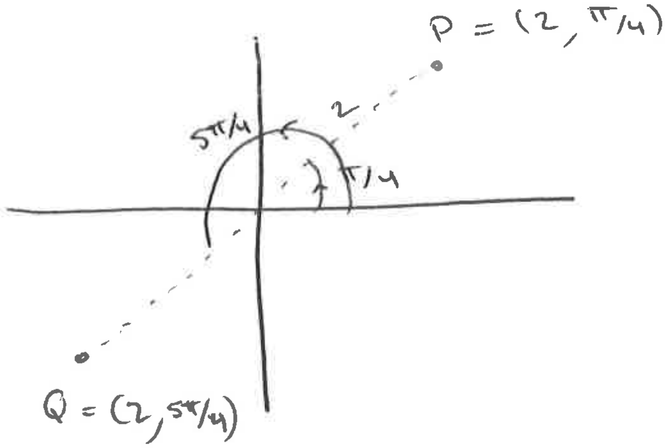


- can also specify P by its polar coordinates (r,θ), where r = distance to origin
  θ = angle made w/ x-axis.

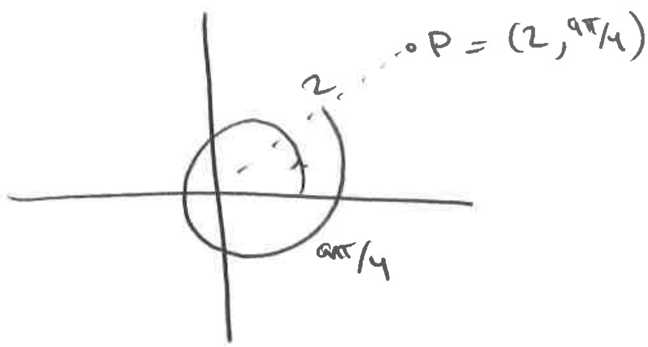


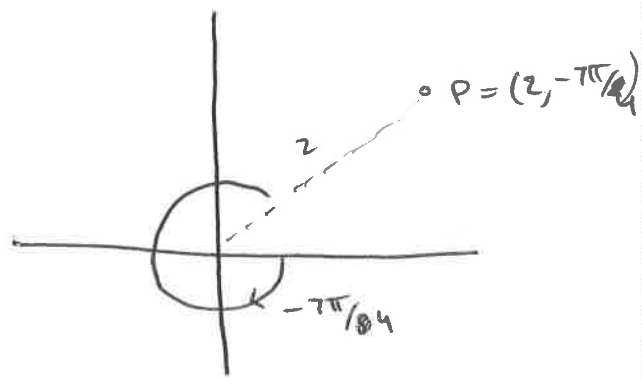
ex: $P = (2, \pi/4)$
$Q = (2, 5\pi/4)$ are shown below:

(ii)



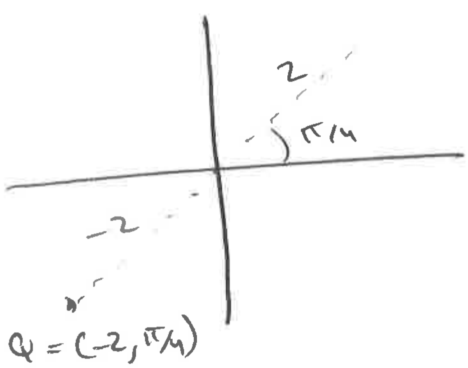
we allow $\theta > 2\pi$ and $\theta < 0$, e.g. P also has coords $(2, 9\pi/4)$ and $(2, -7\pi/4)$





So: polar coords are <u>not</u> unique!

also allow $r < 0$, e.g. $Q = (-2, \pi/4)$

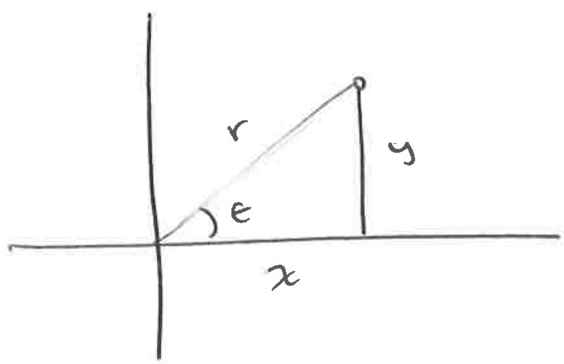


(iii)

To translate:

from polar to rect: use: $x = r\cos\theta$
$y = r\sin\theta$

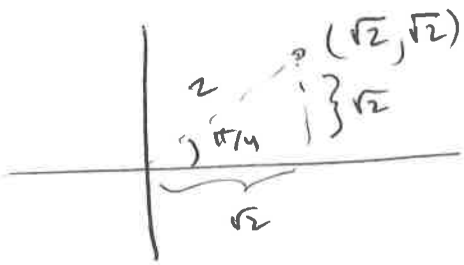
From rect. to polar, use: $r^2 = x^2 + y^2$
$\tan\theta = \dfrac{y}{x}$.



$\longrightarrow$ $x = r\cos\theta$
$y = r\sin\theta$
$x^2 + y^2 = r^2$
$\tan\theta = y/x$

ex: if P has polar coords $(2, \pi/4)$
then P has rectangular coords

$$x = 2\cos(\pi/4) = 2\sqrt{2}/2 = \sqrt{2}$$
$$y = 2\sin(\pi/4) = 2\sqrt{2}/2 = \sqrt{2}$$

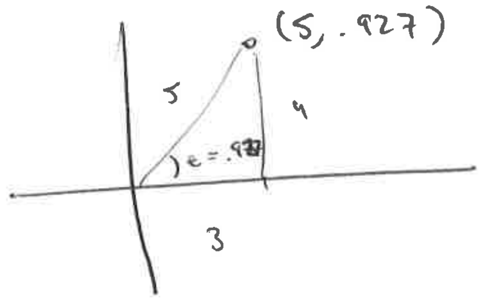


(iv)

if P has rectangular coords $(3,4)$ then P has polar coords given by

$$r^2 = 3^2 + 4^2 = 25 \Rightarrow r = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = .927\ldots$$



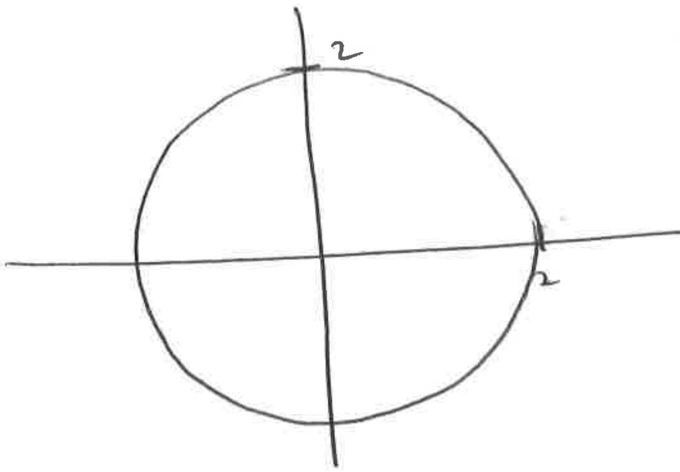
Polar curves

- can also specify curves w/ polar equations
- usually we consider eq'ns of the form $r = f(\theta)$, i.e. when r is a function of θ.
- graph is all polar points (r, θ) where $r = f(\theta)$.

ex: graph the polar curve $r = 2$.

Sol'n: consists of all points (r, θ) where $r = 2$.

(v)



$r = 2$.

to graph more complicated curves $r = f(\theta)$, can take various approaches:

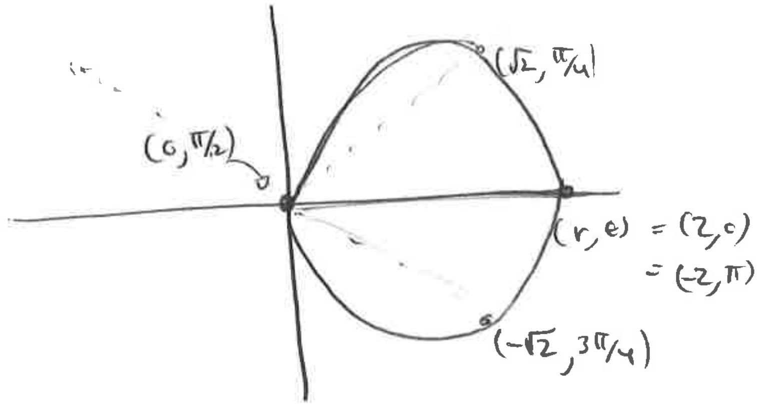
- plot points (not usually effective in itself)
- translate to rectangular coords (doesn't always work)
- use calculus to find points w/ horiz. or vert. tan lines (see second ex below)
- use brain.

ex: graph $r = 2\cos\theta$

Sol'n: can plot some points to start.

(vi)

θ	$r = 2\cos\theta$
0	2
$\pi/4$	$\sqrt{2}$
$\pi/2$	0
$3\pi/4$	$-\sqrt{2}$
π	-2



- only gives a very rough sense of curve.
- in this case we can translate to rectangular coords, but requires some creativity

use: $x = r\cos\theta \qquad x^2 + y^2 = r^2$
$y = r\sin\theta$

to get ~~[illegible]~~ $r = 2\cos\theta$ into only x, y.

$\Rightarrow \cos\theta = \dfrac{x}{r}$

$\Rightarrow r = 2 \cdot \dfrac{x}{r}$

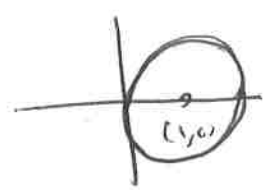
$\Rightarrow r^2 = 2x$

$\Rightarrow x^2 + y^2 = 2x$

$\Rightarrow x^2 - 2x + y^2 = 0$

$\Rightarrow$ ~~[illegible]~~ $x^2 - 2x + 1 + y^2 = 1$

$\Rightarrow (x-1)^2 + y^2 = 1$ circle of radius 1 centered @ (1,0).



(vii)

Using Calculus: to graph a curve $r = f(\theta)$, finding dy/dx at various points more reliable then plotting points randomly and trying to interpolate.

- so we need fmla for $\frac{dy}{dx}$

- using $x = r\cos\theta = f(\theta)\cos(\theta)$

  $y = r\sin\theta = f(\theta)\sin(\theta)$

can view a polar curve $r = f(\theta)$ as a parametric curve (using θ as parameter instead of t)

- we have
$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$
$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

- from before we knew
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \qquad = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

(viii)

ex: a) For the curve $r = 1+\sin\theta$, find those points at which tan line is horizontal or vertical, for $0 \le \theta \le 2\pi$

b) sketch the curve for $0 \le \theta \le 2\pi$.

Sol'n: a) we know
$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$
$dy/d\theta$

$dx/d\theta$

potential

horiz. tan line when $\frac{dy}{dx} = 0$

i.e. $\frac{dr}{d\theta}\sin\theta + r\cos\theta = 0$

but $r = 1+\sin\theta$ so $\frac{dr}{d\theta} = \cos\theta$

so we want: $\cos\theta\sin\theta + (1+\sin\theta)\cos\theta = 0$

i.e. $2\sin\theta\cos\theta + \cos\theta = 0$

i.e. $\cos\theta(1+2\sin\theta) = 0$

i.e. $\cos\theta = 0$ or $\sin\theta = -1/2$

i.e. $\theta = \pi/2,\ 3\pi/2$ $\qquad \theta = 7\pi/6,\ 11\pi/6$

at these θ's we have:

$r(\pi/2) = 1+\sin(\pi/2) = 2$

$r(3\pi/2) = 1+\sin(3\pi/2) = 0$

$r(7\pi/6) = 1+\sin(7\pi/6)$
$= 1/2$

$r(11\pi/6) = 1/2$

(1x)

potential
vert. tan line when $\frac{dy}{dx} = \pm\infty$

Solve for: $\frac{dr}{d\theta}\cos\theta - r\sin\theta = 0$

i.e. $\cos^2\theta - (1+\sin\theta)\sin\theta = 0$

i.e. $\cos^2\theta - \sin\theta - \sin^2\theta = 0$

i.e. $1 - \sin^2\theta - \sin^2\theta - \sin\theta = 0$

i.e. $1 - \sin\theta - 2\sin^2\theta = 0$ " $1-x-2x^2=$

i.e. $(1-2\sin\theta)(1+\sin\theta) = 0$

i.e. $\sin\theta = -1 \Rightarrow 3\pi/2$

or $\sin\theta = 1/2 \Rightarrow \pi/6, 5\pi/6$

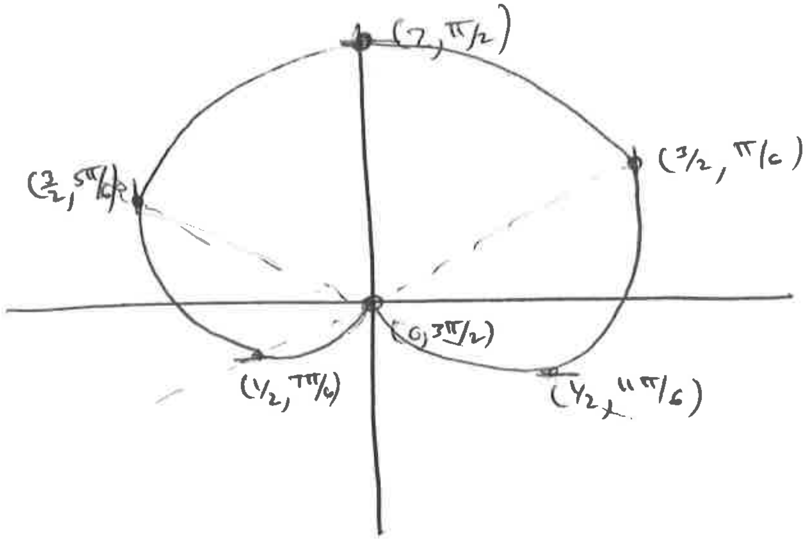
at these pts

$r(3\pi/2) = 1+\sin(3\pi/2) = 0$

$r(\pi/6) = 1+\frac{1}{2} = \frac{3}{2} = r(5\pi/6)$

let's graph these points.

(x)



Observe: @ $(1/2, 7\pi/6)$, $(1/2, 11\pi/6)$, $(2, \pi/2)$

horiz. lines since $\frac{dy}{dx} = \frac{0}{\text{nonzero}}$

@ $(3/2, 5\pi/6)$, $(3/2, \pi/6)$

vert lines since $\frac{dy}{dx} = \frac{\text{nonzero}}{0}$

but @ $(0, 3\pi/2)$ unclear

since $\frac{dy}{dx} = \frac{0}{0}$.

we use L'hospital:

$$\lim_{\theta \to 3\pi/2} \frac{dy}{dx} = \lim_{\theta \to 3\pi/2} \frac{(\cos\theta)(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$$= \underbrace{\lim_{\theta \to 3\pi/2} \left(\frac{\cos\theta}{1+\sin\theta}\right)}_{\frac{0}{0}} \underbrace{\left(\lim_{\theta \to 3\pi/2} \frac{1+2\sin\theta}{1-2\sin\theta}\right)}_{\frac{-1}{3}}$$

(xi) 244

$$= -\frac{1}{3} \lim_{\theta \to 3\pi/2} \frac{\cos\theta}{1+\sin\theta}$$

$$\overset{L'H}{=} -\frac{1}{3} \lim_{\theta \to 3\pi/2} \frac{-\sin\theta}{\cos\theta} = -\frac{1}{3}(\infty) = -\infty.$$

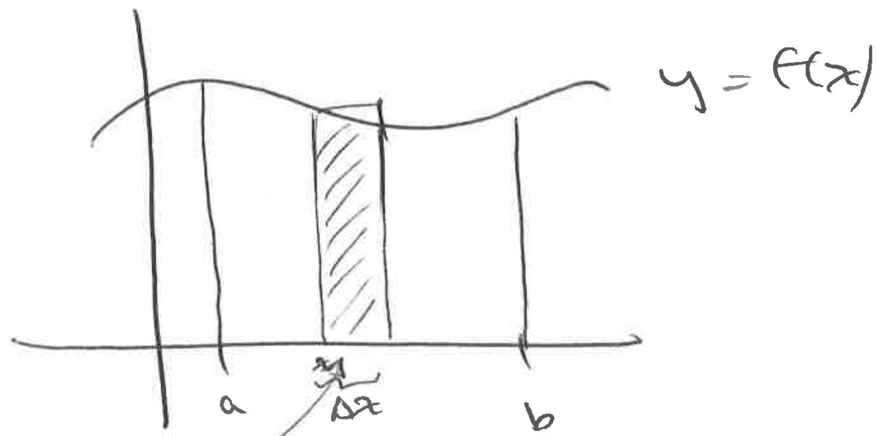
So there is a vertical line @ this pt

Integration and Length in polar coords

integration in rectangular coords:

to find area under $y = f(x)$
between $x = a$ and $x = b$

first approximate:



segment area

$$\approx f(x)\,\Delta x$$

total area

$$\approx \sum f(x)\,\Delta x$$

then take limit:

$$\text{area} = \int_a^b f(x)\,dx$$

$$= \int_a^b y\,dx$$

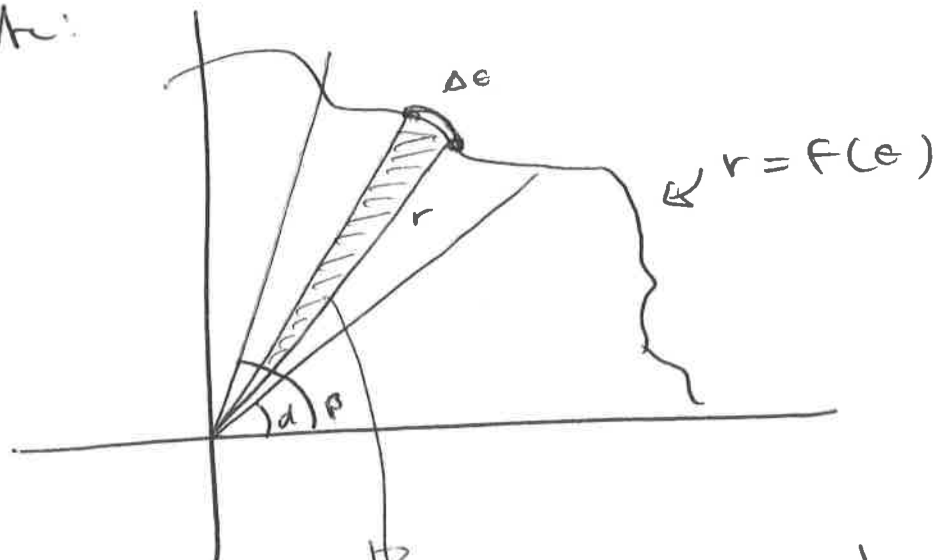
(ii)

In polar coords:

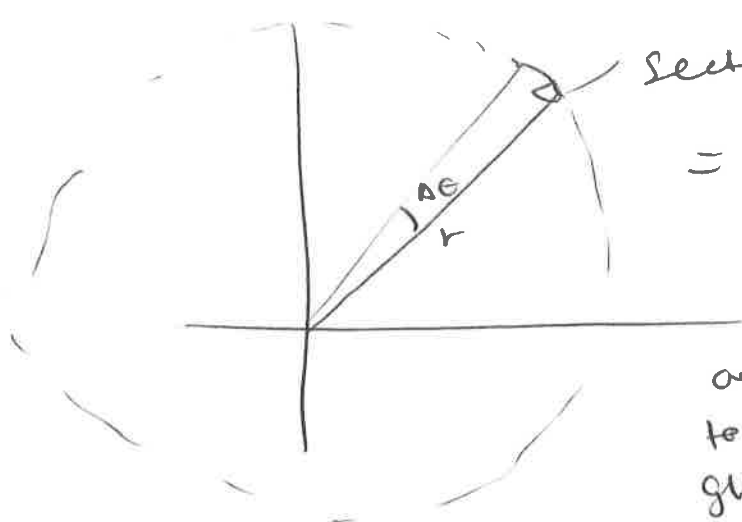
to find area bounded by polar curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$

First:

approximate:



approx this area by a <u>circular</u> <u>sector</u>



sector area $= \frac{1}{2} r^2 \Delta\theta$

(Why: entire area of circle corresponds to $\theta = 2\pi$, which gives Area $= \frac{1}{2} r^2 2\pi = \pi r^2$)

(iii)

So area bounded by $r = f(\theta)$ between $\theta = \alpha, \beta$ is approx:

$$\sum \frac{1}{2} r_i^2 (\Delta\theta)$$

$$= \sum \frac{1}{2} (f(\theta))^2 \Delta\theta$$

So **exact** **area** (taking limit) is:

$$\int_\alpha^\beta \frac{1}{2} r^2 \, d\theta$$

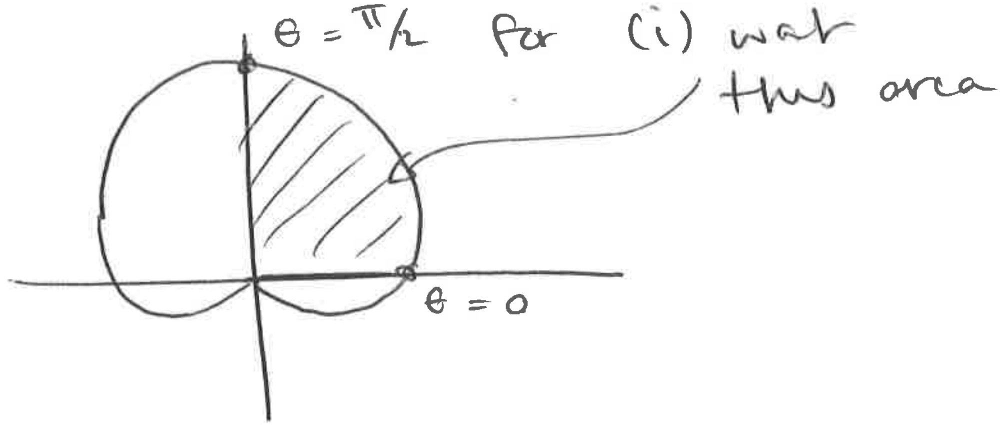
$$= \int_\alpha^\beta \frac{1}{2} (f(\theta))^2 \, d\theta$$

ex: Find the area enclosed by the cardioid $r = 1 + \sin\theta$)

(i) in first quadrant

(ii) overall.

(iv) we know $r = 1+\sin\theta$ looks like: (248)



(i)

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (1+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 + 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

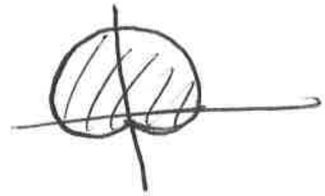
$$= \frac{1}{2}\Big[\theta - 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\Big]_0^{\pi/2}$$

$$= \frac{1}{2}\Big[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\Big]_0^{\pi/2}$$

$$= \frac{1}{2}\Big[\Big(\frac{3}{2}\frac{\pi}{2} - 2\underbrace{\cos\frac{\pi}{2}}_{0} - \frac{1}{4}\underbrace{\sin\pi}_{0}\Big) - \Big(0 - 2\underbrace{\cos 0}_{1} - \frac{1}{4}\underbrace{\sin 0}_{0}\Big)\Big] = \frac{1}{2}\Big(\frac{3\pi}{4} + 2\Big) = \boxed{\frac{3\pi}{8} + 1}$$

(v)

(ii) cycle thru entire cardioid once over $0 \le \theta \le 2\pi$.



So $A = \int_0^{2\pi} \frac{1}{2}(1+\sin\theta)^2 \, d\theta$

$= \vdots$

$\frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[\left(\left(\frac{3}{2}2\pi\right) - 2 - 0\right) - (0 - 2 - 0)\right]$

$= \frac{1}{2}\left[3\pi\right]$

$= 3\pi/2$ ✓

<u>ex</u>: Find area of region <u>inside</u> the circle $r = 3\sin\theta$ and <u>outside</u> the cardioid $r = 1+\sin\theta$.

<u>Sol'n</u>: First: why is $r = 3\sin\theta$ a circle? can translate to rectangular coordinates using $x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$

(vi)

$\Rightarrow \sin\theta = \frac{y}{r}$

So $r = 3\frac{y}{r}$

$\Rightarrow r^2 = 3y$

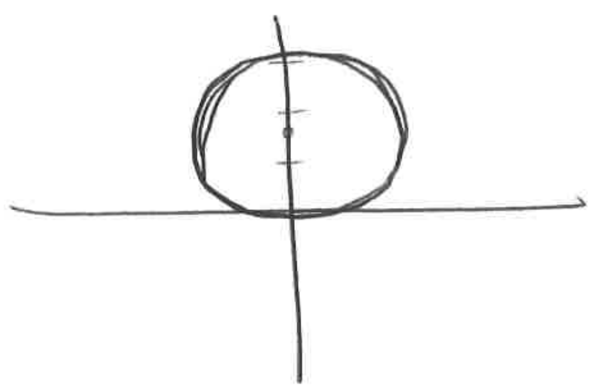
$\Rightarrow x^2 + y^2 = 3y$

$\Rightarrow x^2 + y^2 - 3y = 0$

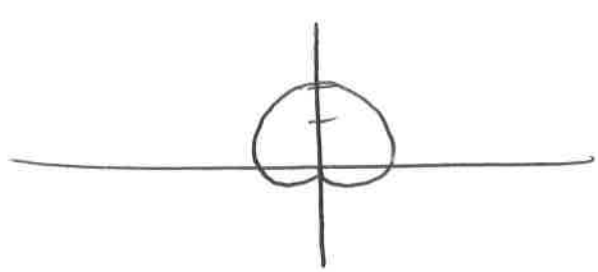
$\Rightarrow x^2 + y^2 - 3y + (\frac{3}{2})^2 = (\frac{3}{2})^2$

$\Rightarrow x^2 + (y - \frac{3}{2})^2 = (\frac{3}{2})^2$

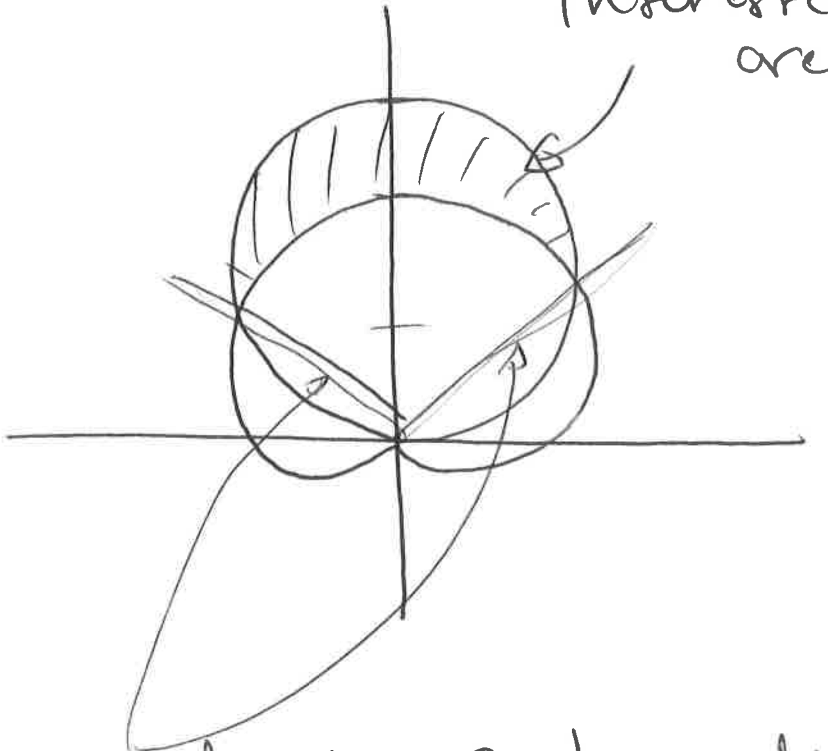
So graph of $r = 3\sin\theta$ is:



Cardioid peaks @ (0,2)



(vii)



need to find angles of intersection.

intersect when:

$$3\sin\theta = 1+\sin\theta$$

$$\Rightarrow \quad 2\sin\theta = 1$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \pi/6,\ 5\pi/6$$

to find area between: subtract area of cardiod from area of circle over $\pi/6 \leq \theta \leq 5\pi/6$

(viii)

(252)

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2\,d\theta - \int_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2\,d\theta$$

$\vdots$ a big mess

$= \pi$.