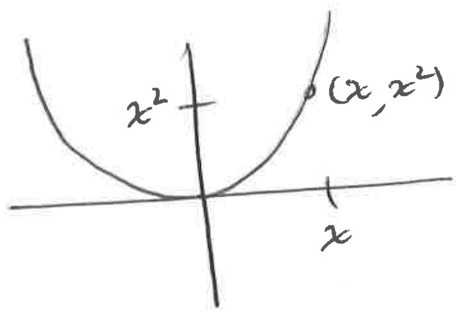
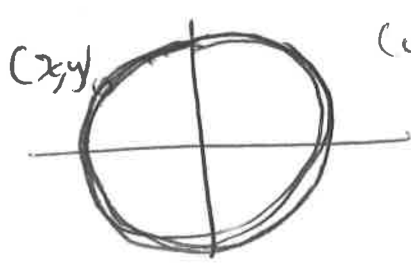
Ch. 10 Calculus of curves.

- Given a (cts) function $f(x)$, graph of $y = f(x)$ is a curve in xy-plane
- consists of all points (x, y) s.t. $y = f(x)$
- e.g. graph of $y = x^2$ consists of all points (x, x^2) for x in $\mathbb{R}$.



- not all curves are of this form
- e.g. the unit circle is not the graph of any function (Fails VLT) but can still be described by eq'n $x^2 + y^2 = 1$



(worst circle ever drawn)

graph consists of all points (x, y) s.t. $x^2 + y^2 = 1$.

② - Another way to define curves: define both coords x and y as functions of a third variable t.

$$x = x(t)$$
$$y = y(t).$$

called a <u>parametrized curve</u>

- think of t as a time variable: at time t a particle is @ $(x(t), y(t))$ - sweeps out curve as t moves forward.

<u>Ex</u>: define $x = t-1$, $y = t^2$. Sketch the parametrized curve.

<u>Sol'n</u>: one way: plot some points!

$t = 0 \rightarrow x = -1 \quad y = 0$

$t = 1 \rightarrow x = 0 \quad y = 1$

$t = 2 \rightarrow x = 1 \quad y = 4$

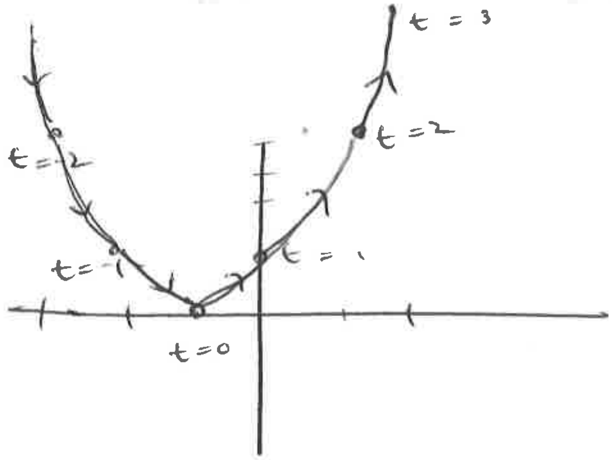
$t = 3 \rightarrow x = 2 \quad y = 9$

$t = -1 \rightarrow x = -2 \quad y = 1$

$t = -2 \rightarrow x = -3 \quad y = 4$

$t = -3 \rightarrow x = -4 \quad y = 9$

③



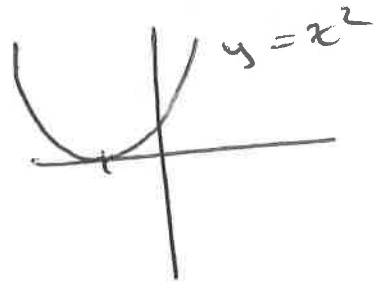
Looks like: parabola w/ vertex @ $(-1,0)$

Note: with param. curves there is a <u>direction</u>, in this case we flow left to right as t increases.

another way: "eliminate the parameter t"
i.e. get eq'n in x and y describing curve.

$x = t-1 \Rightarrow t = x+1$

$y = t^2 \Rightarrow y = (x+1)^2$



Notes: ① work shows: <u>if</u> (x,y) is a pt on parametrized curve, <u>then</u> it also lies on parabola $y=(x+1)^2$

- not necessarily true that <u>every</u> pt on parabola $y=(x+1)^2$ is attained (though in this case this is true)

② in passing to eq'n $y=(x+1)^2$ we lose info about direction of flow as t increases.

④ ex. graph curve: $x = \cos(\pi t)$
$y = \sin(\pi t)$

① for t in $[-1/2, 1/2]$

② for all t.

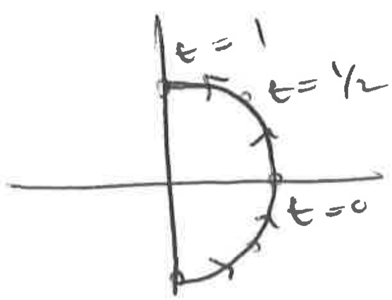
Sol'n $t = -1/2 \Rightarrow x = \cos(-\pi/2) = 0$
$y = \sin(-\pi/2) = -1$

① $t = -1/4$ $x = \frac{\sqrt{2}}{2}$
$y = -\frac{\sqrt{2}}{2}$

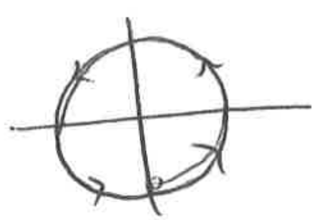
$t = 0$ $x = 1$
$y = 0$

$t = 1/4$ $x = \sqrt{2}/2$
$y = \sqrt{2}/2$

$t = 1/2$ $x = 0$
$y = 1$



② continuing as $t \to \infty$ we see



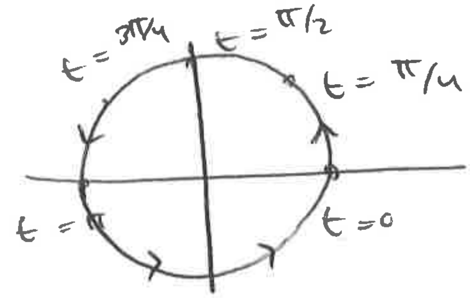
(5) if we elim parameter:

$$x^2 + y^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1$$

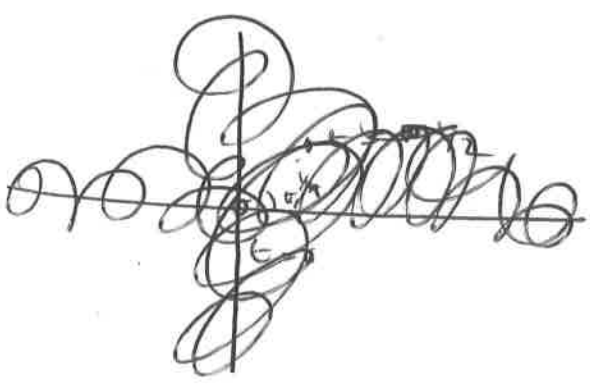
So our curve lies on $x^2+y^2=1$ (as we've seen). ✓

ex: a curve can have multiple parametrizations.

Consider $x = \cos(t)$
$y = \sin(t)$



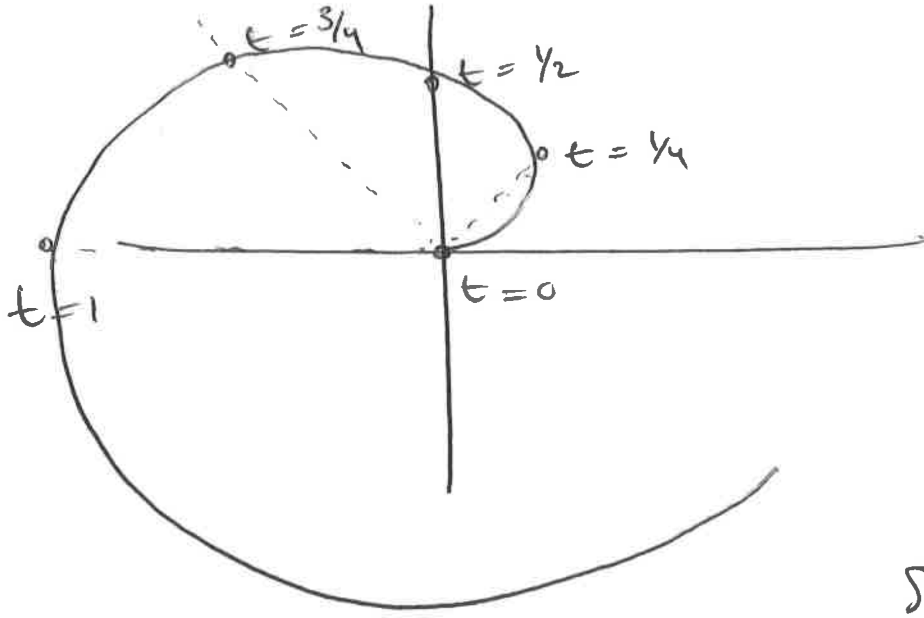
ex: graph $x = t \cdot \cos(\pi t)$
$y = t \cdot \sin(\pi t)$



→ in general: $(r\cos\theta, r\sin\theta)$ is the point on circle of radius r at angle θ

⑥ 225

- for us: @ time t at $(t\cos\pi t, t\sin\pi t)$
So our "radius" is t, angle is πt.



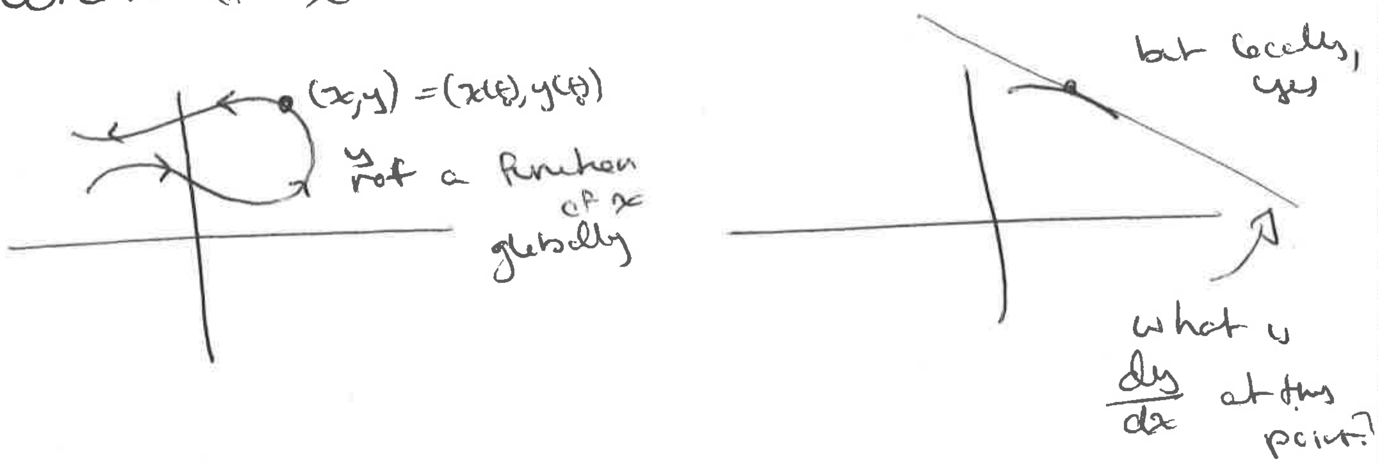
spiral!

- not easy (or useful) to eliminate t in this case.

(7) <u>10.2</u> Calculus of parametric curves.

- we have seen: a parametric curve $x = x(t)$ $y = y(t)$ may not be graph of any function of x (can fail VLT).

- however: if we only look at a piece of the graph of curve near a pt $(x,y) = (x(t), y(t))$, may look like a function of x



- at such a pt can ask: what is slope of tan line, i.e. what is $\frac{dy}{dx}$?

- if we think of $y = y(x)$ as a function of x locally, then since y, x also both functions of t, we have by chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

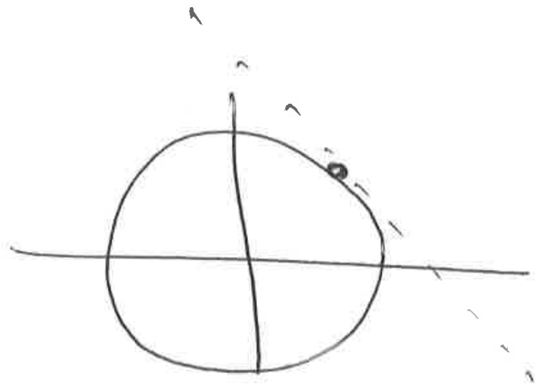
if $dx/dt \neq 0$.

⑧ 227

ex: find the slope of the tan line to the circle $x = \cos(t)$ at the pt
$y = \sin(t)$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

soln:



notice: $x = y = \sqrt{2}/2$ when $t = \pi/4$

Slope is $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$\frac{dy}{dt} = \cos t \qquad \frac{dx}{dt} = -\sin t$

@ $\pi/4$ we have $\frac{dy}{dt} = \cos(\pi/4) = \sqrt{2}/2$

$\frac{dx}{dt} = -\sin(\pi/4) = -\sqrt{2}/2$

so $\frac{dy}{dx} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$.

Ex: Consider the curve C defined by

$$x = t^2$$
$$y = t^3 - 3t$$

(a) Show that C has two tangents @ $(3,0)$, and find their slopes.

Sol'n: observe: $y = 0 \Rightarrow t^3 - 3t = 0$

$\Rightarrow t(t^2 - 3) = 0$

$\Rightarrow t = 0$ or $t = \pm\sqrt{3}$

$\Rightarrow$ curve cross x-axis at these t's.

@ $t = 0$, $x = 0$

@ $t = \sqrt{3}$ or $-\sqrt{3}$ $x = 3$

$\hookrightarrow$ Curve hits $(3,0)$ at $t = \pm\sqrt{3}$

So crosses itself at this pt.

we have: $y'(t) = \dfrac{dy}{dt} = 3t^2 - 3$

$x'(t) = \dfrac{dx}{dt} = 2t$

so $\dfrac{dy}{dx} = \dfrac{3t^2 - 3}{2t}$

at $t = \sqrt{3} \rightarrow \dfrac{dy}{dx} = \dfrac{6}{2\sqrt{3}} = \dfrac{3}{\sqrt{3}} = \sqrt{3} \approx 1.7$

at $t = -\sqrt{3} \rightarrow \dfrac{dy}{dx} = \dfrac{6}{-2\sqrt{3}} = -\sqrt{3} \approx -1.7$

(ii)

22a

$\frac{dy}{dx} = 0$

(b) Find points on curve w/ horizontal or vertical tangent lines

$\frac{dy}{dx} = \pm\infty$

Sol'n <u>horizontal</u>: $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{3t^2-3}{2t} = 0$$

$$\Rightarrow 3t^2 - 3 = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

@ $t = 1$, $x = 1$ $y = -2$

@ $t = -1$, $x = 1$, $y = 2$

So horiz. tangents @ $(1,-2)$ and $(1,2)$

<u>vertical</u>: $\frac{dy}{dx} = \pm\infty$ i.e.

$$\frac{3t^2-3}{2t} = \pm\infty \quad \text{i.e.}$$

$$2t = 0 \Rightarrow t = 0.$$

@ $t = 0$, $x = 0$, $y = 0$

So vertical tangent @ $(0,0)$.

(iii) 230

We can also find a formula for second deriv of y wrt x in terms of t, by subbing $\frac{dy}{dx}$ in for y in our formula for the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

(c) Find where curve C above is concave up $\frac{d^2y}{dx^2} \geq 0$ and concave down $\frac{d^2y}{dx^2} \leq 0$

Sol'n: first we find:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{3t^2-3}{2t}\right)}{2t}$$

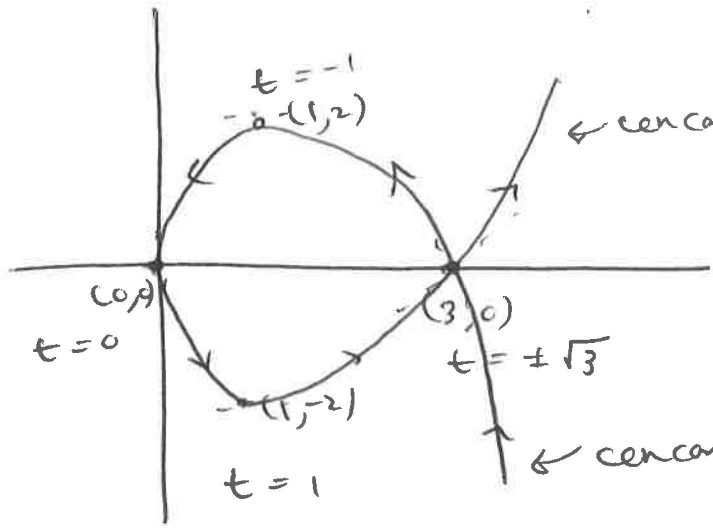
$$= \frac{\frac{2t(6t)-(3t^2-3)2}{(2t)^2}}{2t}$$

$$= \frac{12t^2-6t^2+6}{(2t)^3}$$

$$= \frac{6t^2+6}{8t^3} = \frac{3t^2+3}{4t^3}$$

(iv) Since numerator always positive, second deriv is > 0 when $4t^3 > 0$ i.e. $t > 0$
< 0 when $4t^3 < 0$ i.e. $t < 0$

(d) Sketch curve.



Arc Length

Theorem If a curve C is described by

$$x = f(t)$$
$$y = g(t)$$

and C does not overlap itself (except perhaps at isolated points) for $\alpha \le t \le \beta$ then the length of C over the interval $\alpha \le t \le \beta$ is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(v)

Pf :- Can be derived in a similar way to our previous arc length formula

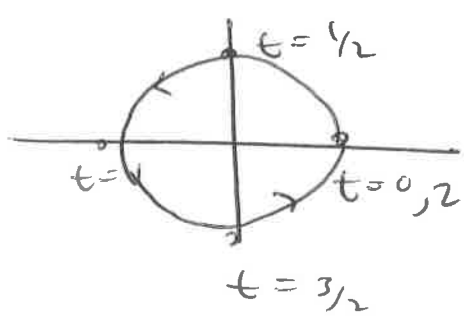
$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$ for curve $y = f(x)$ over $a \le x \le b$.

- see back for details.

ex: Find the circumference of the unit circle, using the parametrization

$$x = \cos(\pi t)$$
$$y = \sin(\pi t).$$

sol'n: this parametrization traverses the circle exactly once as t goes from 0 to 2.



we have: $$\frac{dx}{dt} = -\pi \sin(\pi t)$$
$$\frac{dy}{dt} = \pi \cdot \cos(\pi t)$$

(vi) Sol

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{\pi^2 \sin^2(\pi t) + \pi^2 \cos^2(\pi t)}\, dt$$

$$= \int_0^2 \sqrt{\pi^2}\, dt$$

$$= \int_0^2 \pi\, dt$$

$$= \pi t \Big|_0^2 = 2\pi.$$ ✓

notice: if we integrate from $t = 0$ to $t = 4$

we get $\int_0^4 \pi\, dt$

$= 4\pi$

since this corresponds to the curve that traverses the circle twice.