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Sometimes we can avoid using Taylor's thm by instead using alt. series. estimate to bound $R_N(x)$.

ex. We can approx. $\sin(x) = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

by $x - \frac{x^3}{3!} + \frac{x^5}{5!}$

What is max error in this approx'n for x in $[-.3, .3]$?

Sol'n. Since the Maclaurin series for $\sin(x)$

$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

is alternating, we know that for a fixed x the diff between

$x - \frac{x^3}{3!} + \frac{x^5}{5!}$

and the entire series is at most

$\left|\frac{x^7}{7!}\right|$

For x in $[.3, .3]$, i.e. for x s.t. $|x| \leq .3$ this is at most

(8)

Res

$$\left|\frac{.3^7}{7!}\right| \approx 4.3 \times 10^{-8}$$

so: $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is a very good approx'n to $\sin(x)$ on $[-.3, .3]$

$\sin$ on $[-.3, .3]$

eg. $\sin(12°) = \sin(\pi/15) = .2079116908 1...$

first place of diff

where as

$$(\pi/15) - \frac{(\pi/15)^3}{3!} + \frac{(\pi/15)^5}{5!} = .2079116943 2..$$

Ch. 11 Review:

Sequences: a sequence a_n is an infinite list of numbers, e.g.

$1/2, 1/4, 1/8, \ldots$ $\quad (a_n = \frac{1}{2^n})$

$3, 5, 7, \ldots$ $\quad (a_n = 2n+1)$

$1, -1, 1, -1, \ldots$ $\quad (a_n = (-1)^{n+1})$

are sequences (usually start at $n=1$, but not necessary).

$\lim_{n\to\infty} a_n = L$ means a_n gets arbitrarily close to L as n gets larger (formal def'n in terms of ε's).

Say: a_n converges to L if this happens.

e.g.

$1/2, 1/4, 1/8, \ldots$ converges to 0

$3, 5, 7, \ldots$ diverges (to ∞)

$1, -1, 1, -1, \ldots$ diverges (oscillates)

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Methods to show a_n converges / diverges:

(1) limit laws: e.g. consider sequence

$$a_n = \frac{n^2+1}{2n^2+n} \qquad \left(\tfrac{2}{3}, \tfrac{5}{10}, \tfrac{10}{21}, \ldots \right)$$

don't forget: fmla for a_n is still just a description for a list of numbers.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{2n^2+n} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{2+\frac{1}{n}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)} = \ldots = \frac{1}{2}.$$

(2) use following thm (+ L'hospital, often):

if $a_n = f(n)$ and $\lim_{x\to\infty} f(x) = L$

then $\lim_{n\to\infty} a_n = L$

(first few terms: $1\sin(1), 2\cdot\sin(1/2), \ldots$)

e.g. $\lim_{n\to\infty} n\cdot\sin(1/n)$

$$= \lim_{x\to\infty} x\cdot\sin(1/x)$$

assuming lim exists!

$$= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \quad \text{“}\tfrac{0}{0}\text{”}$$

L'H

$$= \lim_{x\to\infty} \frac{\cancel{-\frac{1}{x^2}}\cdot\cos(1/x)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \cos(1/x) = \cos\left(\lim_{x\to\infty} 1/x\right)$$

$$= \cos(0) = 1 \checkmark$$

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A sequence is a list $a_1, a_2, a_3, \ldots$

To <u>compute</u> $\lim_{n\to\infty} a_n$ or show divergence:

1. limit laws
2. translate to x's use L'hospital
3. inspection (usually to show divergence when a_n oscillates)

To show $\lim_{n\to\infty} a_n$ <u>exists</u> (w/o computing it), can sometimes use:

Thm (monotone convergence): Given a_n a sequence, if either:

1. $a_1 \le a_2 \le a_3 \le \ldots$ and there is C s.t. $C \ge a_n$ for all n; <u>or</u>
2. $a_1 \ge a_2 \ge \ldots$ and there is D s.t. $D \le a_n$ for all n, then

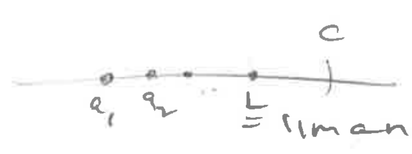
$\lim_{n\to\infty} a_n$ <u>exists</u>

<u>Pf</u>: <u>Idea</u>: in case 1, our sequence

- can't go to ∞ (it's bounded by C)
- can't oscillate (it's increasing

$\Rightarrow$ hence must converge!

might be $\lim_{n\to\infty} a_n < C$



sim idea for 2

⊕

- thm sometimes useful when dealing w/ the sequence of partial sums s_n for some series Σa_n.

- A <u>series</u> is an infinite sum:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + \cdots$$

e.g. $$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2} + \frac{1}{4} + \cdots$$

- the nth partial sum s_n is defined as:

$$s_n = a_0 + a_1 + \cdots + a_n$$

- e.g. for $\sum_0^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2} + \frac{1}{4} + \cdots$

we have

$$s_0 = \frac{1}{2}$$

$$s_1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$s_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

$$\vdots$$

Note: $s_0, s_1, s_2, \ldots$ is a <u>sequence</u>, the terms of which better and better approximate $\sum_0^{\infty} a_n$.

③

Tue

by def'n, $\lim_{n\to\infty} s_n \overset{\text{if lim exists}}{=} \sum_0^\infty a_n$

to be clear: $\sum_0^\infty a_n = L$

means $\lim_{n\to\infty} s_n = L$.

e.g. for $\sum_0^\infty \frac{1}{2^{n+1}}$ we saw $s_n = \frac{2^n-1}{2^n}$

So that $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n-1}{2^n} = \lim_{n\to\infty} 1-\frac{1}{2^n}$

$= 1$

Hence $\sum_0^\infty \frac{1}{2^{n+1}} = \frac{1}{2} + \frac{1}{4} + \cdots = 1$

For a series $\sum_0^\infty a_n$, can also consider the sequence of terms a_n

e.g. For $\sum_0^\infty \frac{1}{2^{n+1}} = \frac{1}{2} + \frac{1}{4} + \cdots$

sequence of terms is $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

Finding the sum $\sum_0^\infty a_n$ is about finding $\lim_{n\to\infty} s_n$, not $\lim_{n\to\infty} a_n$.

Series tests do not apply to finding $\lim_{n\to\infty} a_n$.

But there is a relationship between $\lim a_n$ and $\sum a_n$.

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(Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_0^\infty a_n$ diverges (i.e. $\lim_{n\to\infty} s_n$ diverges).

<u>ex</u>: $\sum_1^\infty \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots$

diverges because $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.

<u>Tests for convergence of $\sum a_n$</u>

(1) Geo. series test: $\sum_1^\infty a r^{n-1}$ converges iff $|r|<1$ (if and only if)

e.g. $\sum_1^\infty 2(\frac{1}{3})^n = 2 + \frac{2}{3} + \frac{2}{9} + \cdots$ converges $\frac{1}{3} < 1$

$\sum_1^\infty 2 \cdot 3^n = 2 + 6 + 18 + \cdots$ diverges $3 > 1$.

(2) p-series test: $\sum_1^\infty \frac{1}{n^p}$ converges iff $p > 1$.

e.g. $\sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$ converges

$\sum_1^\infty \frac{1}{\sqrt{n}}$ diverges

(3) integral test: if $a_n = f(n)$ where $f(x)$ is positive and decreasing on $[1,\infty)$ then:

$\sum_1^\infty a_n$ converges iff $\int_1^\infty f(x)\,dx$ converges

e.g. $\sum_2^\infty \frac{1}{n\sqrt{\ln(n)}} = \frac{1}{2\sqrt{\ln(2)}} + \frac{1}{3\sqrt{\ln(3)}} + \cdots$ diverges

because $\int_2^\infty \frac{1}{x\sqrt{\ln(x)}} = \infty$ (check...)

(4) Comparison (note: applies to series $\sum a_n$ $\sum b_n$ w/ positive terms)

Direct: • $\sum_1^\infty \frac{n}{n^3+1}$ converges since

$$\frac{n}{n^3+1} \le \frac{n}{n^3} = \frac{1}{n^2}$$

and $\sum_1^\infty \frac{1}{n^2}$ converges

• $\sum_1^\infty \frac{n^2+1}{n^3}$ diverges since

$$\frac{n^2+1}{n^3} \ge \frac{n^2}{n^3} = \frac{1}{n}$$

and $\sum_1^\infty \frac{1}{n}$ diverges

limit: if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$

then $\sum a_n$ converges iff $\sum b_n$ does.

• $\sum_2^\infty \frac{n}{n^3-1}$ converges because:

(hint: use this for #7)

$$\lim_{n\to\infty} \frac{n/n^3-1}{1/n^2} = \lim_{n\to\infty} n^3/n^3-1 = 1$$

and $\sum_2^\infty \frac{1}{n^2}$ converges.

(5) Alt series test for $\sum_0^\infty (-1)^n b_n$ $(b_n \ge 0)$

says: if

① $\lim_{n\to\infty} b_n = 0$

② $b_{n+1} \le b_n$ for every n

then $\sum_0^\infty (-1)^n b_n$ converges.

⑥

e.g. $\sum_{1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$ converges

since $\lim_{n\to\infty} \frac{1}{n} = 0$

$\frac{1}{n+1} < \frac{1}{n}$ for all n

Also useful: Alt series estimation: if $\sum_{1}^{\infty} (-1)^{n+1} b_n$ is an alt series satisfying ① and ② then

$$s_n = b_1 - b_2 + b_3 - \cdots \pm b_n$$

is within b_{n+1} of entire series $\sum (-1)^{n+1} b_n$

(i.e. $|R_n| \le b_{n+1}$)

e.g. $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ is w/in $\frac{1}{4}$ of $\sum_{1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(6) ratio / root tests

Def'n $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges

Fact: absolute convergence $\Rightarrow$ convergence (but not the other way around)

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ex Sps $\sum_0^\infty a_n$ $= a_0 + a_1 + \cdots$ is a series and consider $\sum |a_n|$ $= |a_0| + |a_1| + \cdots$

Let $S_n = |a_0| + |a_1| + \cdots + |a_n|$ be the nth partial sum and assume

$$S_n \le 1 \quad \underline{\text{for}}\ \underline{\text{all}}\ \underline{n}$$

Does $\sum a_n$ converge?

A: yes: the sequence S_n is

- increasing (why: $S_{n+1} = S_n + |a_{n+1}| \ge S_n$)
- bounded (by 1)

hence converges by monotone convergence
(i.e. $\lim_{n\to\infty} S_n = L$ exists)

hence $\sum |a_n|$ converges (by def'n)

hence $\sum a_n$ converges ✓

(Note: ratio / root tests give <u>absolute</u> convergence)

e.g. $\sum_0^\infty \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \cdots$ converges

since $\lim_{n\to\infty} \left| \frac{1/(n+1)!}{1/n!} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

as does $\sum_0^\infty \frac{1}{n^n}$ since $\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}\right|} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$.

(8) <u>Power series</u>: a powers series (centered at $x=a$) is a formal sum $\sum_0^\infty c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

e.g. $\sum_0^\infty \frac{(-1)^n}{n+1} x^n = 1 - \frac{1}{2}x + \frac{1}{3}x^2 + \cdots$ is a power series

<u>different</u> than a series

if we specify an x, becomes a series.

E.g. if $x=2$ in above we get:

$$\sum_0^\infty \frac{(-1)^n}{n+1} 2^n = 1 - \frac{1}{2}\cdot 2 + \frac{1}{3}\cdot 4 - \frac{1}{4} 8 + -$$

(diverges)

Big q: for <u>which</u> x's does $\sum_0^\infty c_n(x-a)^n$ <u>converge</u>?

Use ratio/root test to answer.

e.g. for series above: $\sum_0^\infty \frac{(-1)^n}{n+1} x^n$

we have: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{n+1}/n+2}{(-1)^n x^n / n+1} \right|$

$$= \lim_{n\to\infty} |x| \cdot \frac{n+1}{n+2}$$

$$= |x|$$

so by ratio test, series converges if $|x|<1$, i.e. x in $(-1,1)$. Diverges if $|x|>1$.

Have to check @ $x=-1, 1$ by hand. (you try)

(9) (26)

On its interval of convergence, a power series defines a function.

$$\sum_0^\infty c_n(x-a)^n = f(x) \quad \text{on} \quad (a-R, a+R)$$

(think of $f(x)$ as "infinite polynomial")

integration + differentiation behave as you expect:

$$\text{if} \quad f(x) = \sum_0^\infty c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$f' = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$\int f\,dx = c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \cdots + C$$

Amazing fact: many well-known functions can be rep'd as power series (at least on parts of their domain)

$$\frac{1}{1-x} = \sum_0^\infty x^n \qquad x \text{ in } (-1,1)$$

$$e^x = \sum_0^\infty \frac{1}{n!}x^n \qquad \text{for all } x$$

$$e^{-x^2} = \sum_0^\infty \frac{1}{n!}(-x^2)^n = \sum_0^\infty \frac{(-1)^n}{n!}x^{2n} \qquad \text{all } x$$

(10)

we can use these rep'ns to get others:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_0^\infty x^n$$

$$= \frac{d}{dx}\left(1+x+x^2+\cdots\right)$$

$$= 1+2x+3x^2+\cdots$$

$$= \sum_{1}^{\infty} n x^{n-1}$$

x in $(-1,1)$

or even to get power series rep'n for functions we can't otherwise write:

$$\int e^{-x^2}dx = \int \sum_0^\infty \frac{(-1)^n}{n!} x^{2n} = \sum_0^\infty \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + C$$

can't be written in terms of elem functions!

Taylor series: if $f(x)$ has a power series rep'n around $x=a$, it's given by its Taylor series

$$f(x) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$$

11 e.g. if $f(x) = \sin x$

$$f(0) = \sin(0) = 0$$
$$f'(x) = \cos(0) = 1$$
$$f''(x) = -\sin(0) = 0$$
$$f'''(x) = -\cos(0) = -1$$

So Taylor series at $a=0$

$$\sin(x) = 0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots$$
$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots$$
$$= \sum_0^\infty \frac{1}{(2n+1)!}x^{2n+1}$$ (converges to $\sin(x)$ for all x)

Power series rep'ns useful for <u>estimating</u> functions $f(x)$ by polynomials.

if $f(x) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $(a-R, a+R)$

then the Nth <u>Taylor</u> <u>polynomial</u> is:

$$\sum_0^N \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \ldots + f^{(N)}(a)(x-a)^N$$

we have $f(x) \approx T_N^{a}(x)$ on $(a-R, a+R)$

where $\approx$ gets better the bigger the N and the closer x is to a.

Error given by $R_N(x) = \sum_{N+1}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$

② 2a

Can sometimes bound $R_N(x)$ w/ alt series estimate thm

e.g. $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$

by alt series thm

$x - \frac{x^3}{3!} + \frac{x^5}{5!}$ is always w/in $\left|\frac{x^7}{7!}\right|$ of $\sin(x)$

So e.g. in $(-1,1)$ this poly approx's $\sin(x)$ to w/in $\frac{1}{7!}$.

if alt series doesn't apply, use Taylor's thm to bound $R_N(x)$.