Taylor and Maclaurin series

The big Q: which functions f have power series rep'ns? How do we find them?

So far: started with

$$f(x) = \frac{1}{1-x} = \sum_{0}^{\infty} x^n \quad \text{for } -1<x<1.$$

and found power series for variations of this $f(x)$: $\frac{1}{1+x^2}$, $\ln(1+x)$, $\frac{x}{2+x}$, ... etc.

A new approach: Sps we are given a function $f(x)$ and we assume $f(x)$ has a power series rep'n

$$f(x) = \sum_{0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

on some interval of the form $(a-R, a+R)$ but we don't know the c_n's.

How do we find them?

② Observe: by our differentiation rules for power series, we have:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

so: $f'(a) = c_1 = 1 \cdot c_1 = 1!\, c_1$

$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots$$

so: $f''(a) = 2c_2 = 2!\, c_2$

$$f'''(x) = 6c_3 + 24c_4(x-a) + 120c_5(x-a)^2 + \cdots$$

so: $f'''(a) = 6c_3 = 3!\, c_3$

and in general we see:

$$f^{(n)}(x) = n!\, c_n + (n+1)!\, c_{n+1}(x-a) + \cdots$$

so: $f^{(n)}(a) = n!\, c_n$

We get a formula for c_n!

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This is big news: says if we can find the derivatives $f^{(n)}(a)$ of f at a, we can solve for the coefficients c_n in power series rep'n for f ... assuming such a rep'n exists.

Thm: If $f(x)$ has a power series rep'n at a, i.e. there is $R>0$ such that

$$f(x) = \sum_{0}^{\infty} c_n (x-a)^n \quad \text{if } |x-a|<R,$$

then the coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

So that

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{if } |x-a|<R.$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

— this series is called the Taylor Series for $f(x)$ around $x=a$.

— in case when $a=0$, this series is called the Maclaurin series for f.

Thm says: IF f(x) has a power series rep'n @ $x=a$ then that rep'n is given by the Taylor series.

- Some f's do not have a pow. series rep'n anywhere
- Such f's will not equal their Taylor series anywhere
- we will generally ignore the question of "does this f(x) have a Taylor series expansion at a?"

  and just assume that it does.
- But you should be aware this assumption is being made

  (See book for how to prove a given f converges to its Taylor series).

Ex: Find Maclaurin series for $f(x) = e^x$ and its radius of convergence.

Sol'n By theorem, Maclaurin series given by:

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$$\sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n$$

In this case, $f(x) = e^x$. So $f'(x) = f''(x) = \cdots$
$= e^x$.

Hence $f^{(n)}(0) = e^0 = 1$, for every n.
So the Maclaurin series is:

$$\sum_0^\infty \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

we've checked before: this series converges <u>everywhere</u> (i.e. radius is ∞).

<u>Turns out</u>: can be <u>proved</u> e^x equals its Maclaurin series, i.e.

$$e^x = \sum_0^\infty \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \cdots$$

for every x!

In particular:

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Can use this expression to get approx's for e.

e.g. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708\overline{3}$

whereas $e = 2.718\ldots$

<u>Note</u>: a given function $f(x)$ can have different power series rep'ns around different centers a.

<u>ex</u>: Find Taylor series for e^x around $a = 2$.

<u>Sol'n</u>: by theorem, Taylor series at $a=2$ given by:

$$\sum_{0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

here $f(x) = e^x$, $f^{(n)}(x) = e^x$, so $f^{(n)}(2) = e^2$ for <u>every</u> n. So Taylor series is:

$$\sum_{0}^{\infty} \frac{e^2}{n!}(x-2)^n = e^2 + e^2(x-2) + e^2/2\,(x-2)^2 + e^2/6\,(x-2)^3 + \cdots$$

EZ to check: this series also converges everywhere, and $= e^x$ everywhere

Hence $e^x = \sum_{0}^{\infty} \frac{e^2}{n!}(x-2)^n$ for every x.

(7) <u>ex</u>: Find the Maclaurin series for $f(x) = \sin(x)$, and its radius of convergence.

<u>Sol'n</u>. Series given by:

$$\sum_{0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \cancel{\frac{f(0)}{0!}}^{\,0} + \frac{f'(0)}{1!} x + \cancel{\frac{f''(0)}{2!} x^2}^{\,0} + \cdots$$

we check:

$$\begin{aligned}
f(0) &= \sin(0) = 0 \\
f'(0) &= \cos(0) = 1 \\
f''(0) &= -\sin(0) = 0 \\
f'''(0) &= -\cos(0) = -1 \\
f^{(4)}(0) &= \sin(0) = 0 \\
f^{(5)}(0) &= \cos(0) = 1 \\
&\vdots
\end{aligned}$$

etc.

So series is:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

To find radius, use ratio test:

$$\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \Big/ \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{\overbrace{(2n+1)!}^{1\cdot 2\cdots(2n+1)}}{\underbrace{(2n+3)!}_{1\cdot 2\cdots(2n+1)(2n+2)(2n+3)}} \right|$$

$$= \lim_{n\to\infty} \left| x^2 \cdot \frac{1}{(2n+2)(2n+3)} \right|$$

$$= x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = 0$$, no matter the x.

$\Rightarrow$ series converges everywhere $(R=\infty)$.

Can be proven: $\sin(x)$ = its Maclaurin series where it converges.

Hence $\sin(x) = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x!

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ex. Find Maclaurin series for $f(x) = \cos(x)$.

Sol'n: Two approaches:

- could use def'n $\sum_0^\infty \frac{f^{(n)}(0)}{n!} x^n$

or

- differentiate series for $\sin(x)$.

Let's try the second:

$$\cos(x) = \frac{d}{dx}\sin(x) = \frac{d}{dx} \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{0}^{\infty} \frac{d}{dx} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

diff'e theorem

$$= \sum_{0}^{\infty} (-1)^n \frac{(2n+1)\, x^{2n}}{(2n+1)!}$$

$$= \sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Since series for $\sin(x)$ converges, we knew series for $\cos(x)$ does as well

Hence $\cos(x) = \sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

for all x.

ex: Find Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

Sol'n: ez to check
$f(0) = 1$ and
$f^{(n)}(0) = k(k-1)\cdots(k+n-1)$ for $n \geq 1$

So Maclaurin series is:

(c) $$\sum_0^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_0^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$$

can also check: radius of convergence is 1 (w/ ratio test).

It can be proved $(1+x)^k$ = its Maclaurin series on its interval of convergence

Here: for $|x| < 1$,

when $n=0$ this is 1

$$(1+x)^k = \sum_0^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

the coefficient $\frac{k(k-1)\cdots(k-n+1)}{n!}$ is often denoted $\binom{k}{n}$, ~~[illegible]~~.

So we can rewrite:

when $n=0$ this is 1

$$= \sum_0^{\infty} \binom{k}{n} x^n$$

~~[illegible]~~ This series mainly useful when k is fractional, or negative

See Table 1, pg. 768 for list of useful Taylor/Maclaurin series.

(10.5) ex Find Maclaurin series for $f(x) = \frac{1}{\sqrt{1+x}}$. (ac)

Sol'n: $\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$

$$= \sum_0^\infty \binom{-1/2}{n} x^n$$

above for $|x|<1$

$$= \sum_0^\infty \frac{-\frac{1}{2}(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!} x^n$$

$$= 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots$$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2 2!}x^2 - \frac{1\cdot 3\cdot 5}{2^3 3!}x^3$$

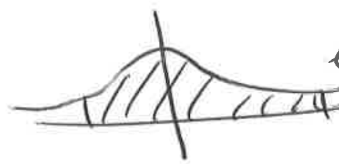
$$= \sum_0^\infty (-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n n!} x^n$$

converges on $(-1,1)$

(11) Using Taylor/Maclaurin series to integrate

<u>ex</u>: Find $\int e^{-x^2}dx$ as a power series and estimate $\int_0^1 e^{-x^2}dx$ to within .001.

<u>Sol'n</u>: the integral $\int e^{-x^2}$ famously cannot be written in terms of elementary functions. It computes the area under the "bell curve" e^{-x^2}.



But we can find a power series for e^{-x^2}, and so for $\int e^{-x^2}dx$ too.

→ could find e^{-x^2} using def'n of Mac series, but instead let's use

$$e^x = \sum_0^\infty \frac{x^n}{n!} \qquad \text{all } x$$

$$\text{so } e^{-x^2} = \sum_0^\infty \frac{(-x^2)^n}{n!}$$

$$= \sum_0^\infty \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

(2) Hence:

$$\int e^{-x^2}\,dx = \int \sum_0^\infty (-1)^n \frac{x^{2n}}{n!}\,dx$$

$$= \sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)\,n!} + C$$

$$= x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + C$$

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so $$\int_0^1 e^{-x^2} = \left[x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots \right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \underset{\substack{\| \\ \frac{1}{9\cdot 4!}}}{\frac{1}{216}} - \cdots$$

$$\approx \overset{a_0}{1} - \overset{a_1}{\frac{1}{3}} + \overset{a_2}{\frac{1}{10}} - \overset{a_3}{\frac{1}{42}} + \overset{a_4}{\frac{1}{216}}$$

$$\approx .7475$$

by alt. series thm:

$$\underset{\substack{\| \\ R_4}}{\text{error}} < b_5 = \frac{1}{11\cdot 5!} = \frac{1}{1320} = .00075 < .001$$

(1)

Operations on Power Series

- Turns out we can add, subtract, multiply, and divide power series just like regular polynomials... though it can get messy.
- Can use to get new power series rep's.
- Sps $\sum_0^\infty c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ converge on $(-R, R)$

  $\sum_0^\infty d_n x^n = d_0 + d_1 x + d_2 x^2 + \cdots$ converge on $(-S, S)$

are power series

Then: if a a real number:

$$a \sum_0^\infty c_n x^n = a(c_0 + c_1 x + c_2 x^2 + \cdots)$$
$$= ac_0 + ac_1 x + ac_2 x^2 + \cdots$$
$$= \sum_0^\infty a c_n x^n \quad \text{converges on } (-R, R)$$

ex: Since $\frac{1}{1-x} = \sum_0^\infty x^n$ on $(-1, 1)$

we have $\frac{2}{1-x} = 2 \sum_0^\infty x^n = \sum_0^\infty 2x^n$

$$= 2 + 2x + 2x^2 + \cdots$$

on $(-1, 1)$

Also: $x^k \sum_0^\infty c_n x^n = x^k(c_0 + c_1 x + c_2 x^2 + \cdots)$

$= (c_0 x^k + c_1 x^{k+1} + c_2 x^{k+2} + \cdots)$

$= \sum_0^\infty c_n x^{n+k}$ converges on $(-R, R)$

ex: $e^x = \sum_0^\infty \frac{1}{n!} x^n$ on $(-\infty, \infty)$

So $x^2 e^x = \sum_0^\infty \frac{1}{n!} x^{n+2}$ on $(-\infty, \infty)$

radius R radius S.

Also: $\left(\sum_0^\infty c_n x^n\right)\left(\sum_0^\infty d_n x^n\right)$

$= (c_0 + c_1 x + c_2 x^2 + \cdots)(d_0 + d_1 x + d_2 x^2 + \cdots)$

$= c_0 d_0 + c_0 d_1 x + c_1 d_0 x + c_0 d_2 x^2 + c_1 d_1 x^2 + c_2 d_0 x^2 + \cdots$

$= c_0 d_0 + (c_0 d_1 + c_1 d_0) x + (c_0 d_2 + c_1 d_1 + c_2 d_0) x^2 + \cdots$

radius of convergence will be the <u>smaller</u> of R, S.

(3)

<u>ex</u>: we know $e^x = \sum_0^\infty \frac{1}{n!}x^n$ on $(-\infty, \infty)$

$$\frac{1}{1-x} = \sum_0^\infty x^n \quad \text{on } (-1,1)$$

so: $$\frac{e^x}{1-x} = e^x\left(\frac{1}{1-x}\right) = \left(\sum \frac{1}{n!}x^n\right)\left(\sum x^n\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= 1 + x + x + x^2 + x^2 + \frac{x^2}{2!} +$$

$$x^3 + x^3 + \frac{x^3}{2!} + \frac{x^3}{3!} + \cdots$$

$$= 1 + 2x + \left(\overset{\frac{1}{0!}+\frac{1}{1!}}{2} + \frac{1}{2!}\right)x^2 + \left(\overset{\frac{1}{0!}+\frac{1}{1!}}{2} + \frac{1}{2!} + \frac{1}{3!}\right)x^3$$

$$+ \cdots + \left(\overset{\frac{1}{0!}+\frac{1}{1!}}{2} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)x^n + \cdots$$

$$= \sum_0^\infty \left(\frac{1}{0!} + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)x^n \quad \text{on } (-1,1)$$

take the <u>smaller</u> radius.

④ Also: can do <u>long division</u> of power series ... but it's a mess so we'll ignore.

↳ can be used, for example, to find series rep'n for $\tan(x)$

$$= \frac{\sin(x)}{\cos x} = \frac{\sum_0^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}}{\sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!}}$$

$$= \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots}{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots}$$

$= ?$ See pg. 770 if interested.

① Taylor Polynomials

Def'n Sps that f(x) is a function w/ a Taylor series rep'n @ $x=a$.

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad \text{for } x \text{ in } (a-R, a+R)$$

We define the Taylor polynomials of f(x) as follows:

$$T_0(x) = f(a)$$
$$T_1(x) = f(a) + f'(a)(x-a)$$
$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$
$$\vdots$$
$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$\vdots$$

that is: the Nth Taylor polynomial is just the sum of the first N terms of f's Taylor series

the point: since f is = its Taylor series on (a-R, a+R), the Taylor poly's will be good approx's to f on that interval.

2 the larger N is, and the closer x is to the center a, the better the approx'n $T_N(x)$ will be for $f(x)$.

ex. We know

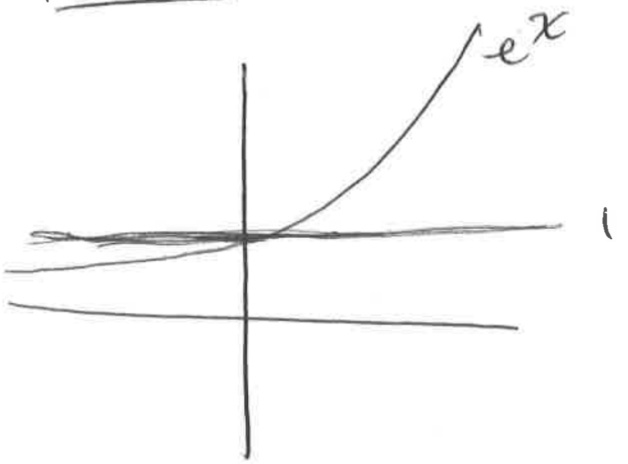
$$e^x = \sum_0^\infty \frac{1}{n!} x^n \quad \text{on } (-\infty, \infty).$$

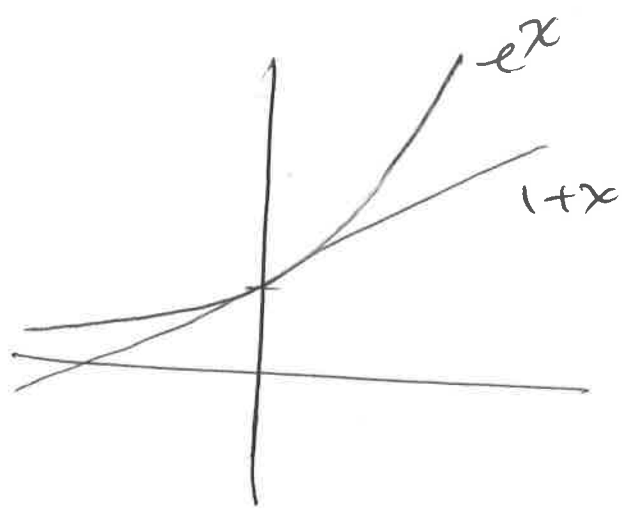
First few Taylor poly's are:

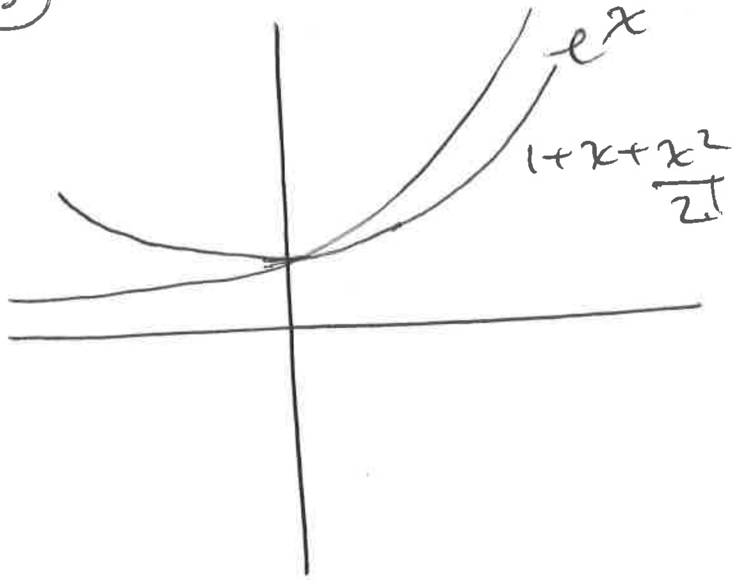
$$T_0(x) = 1$$
$$T_1(x) = 1 + x$$
$$T_2(x) = 1 + x + \frac{x^2}{2!}$$
$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

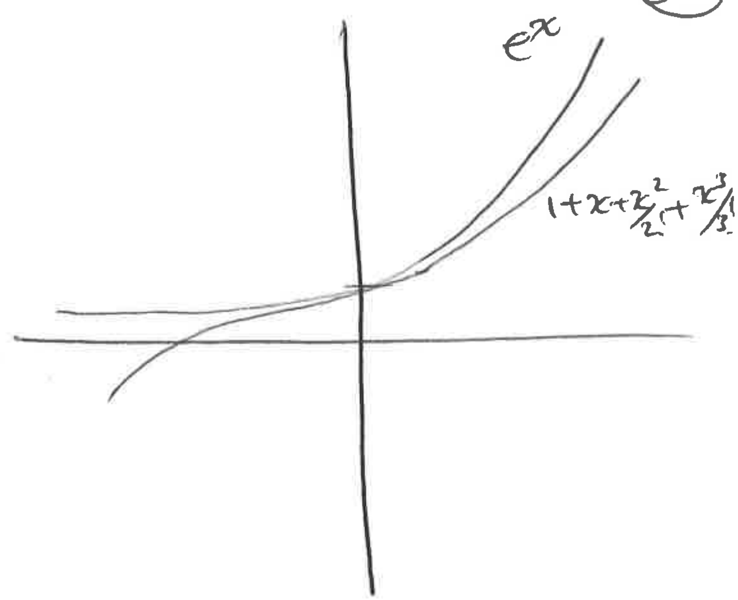
We expect these poly's to be "reasonably" good approx's to e^x ... near $a = 0$.

Picture:









... tho if we zoom out, approx's get worse.

Big Q: How good an approx. is $T_N(x)$ for e^x?

To answer, need a def'n:

Def'n: Given $f(x)$ and the Taylor series $\sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$ ($= f$ on $(a-R, a+R)$) for f @ $x=a$, we define the Nth remainder function

$$R_N(x) = \underbrace{f(x)}_{= \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n} - \underbrace{T_N(x)}_{= \sum_0^N \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

$$= \sum_{N+1}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f^{(N+1)}(a)}{(N+1)!}(x-a)^{N+1} + \frac{f^{(N+2)}(a)}{(N+2)!} + \cdots$$

(4)

- i.e., $R_N(x)$ is just the "tail" of the Taylor series for $f(x)$, beginning @ $N+1$.
- for a given x in $(a-R, a+R)$, $R_N(x)$ tells us how far $T_N(x)$ is from $f(x)$.
- we'd like a theorem that bounds the remainder.

Given by:

<u>Taylor's Theorem</u> Sps $f(x) = \sum_0^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$ on $(a-R, a+R)$. Then if there is a constant M such that $|f^{(N+1)}(x)| \le M$ on $(a-d, a+d)$ where $d \le R$, we have:

$$|R_N(x)| \le \frac{M}{(N+1)!}|x-a|^{N+1} \quad \text{on } (a-d, a+d)$$

<u>Says</u>: if we can bound the $(N+1)$st derivative of f on an interval $(a-d, a+d)$ around a, then we can bound size of remainder on that interval.

↳ won't prove.

(5)

ex (1) Approximate $f(x) = \sqrt[3]{x} = x^{1/3}$ by $T_2(x)$ around $a = 8$.

(2) How accurate is this approximation for x in $(7, 9)$?

$8-1 \quad 8+1$

Sol'n: (1) Let's find $T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2$

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \quad f'(8) = \frac{1}{3}8^{-2/3} = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-5/3} \quad f''(8) = -\frac{2}{9}8^{-5/3}$$

$$= -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-8/3}$$

need this to bound $R_2(x)$

$-1/144/2$

so $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

$\approx \sqrt[3]{x}$ near $a = 8$.

(2) By Taylor's thm, if we can find M s.t. $|f'''(x)| \le M$ for x in $(7,9)$

(6)

then we can bound $R_2(x)$ on this interval.

Observe: $x \geq 7$ on $(7,9)$ so

$x^{8/3} \geq 7^{8/3}$ on " " so

$\frac{1}{x^{8/3}} \leq \frac{1}{7^{8/3}}$ on " " so

$$f'''(x) = \frac{10}{27}\frac{1}{x^{8/3}} \leq \frac{10}{27}\cdot\frac{1}{7^{8/3}} < .0021 \text{ on } (7,9)$$

Taylor says:

$$|R_2(x)| \leq \frac{.0021}{3!}|x-8|^3 \quad \text{for } x \text{ in } (7,9)$$

$$\leq \frac{.0021}{3!}1^3 \quad \text{for } x \text{ in } (7,9)$$

$$= .0004.$$

This means: if we plug in on x in $(7,9)$ into $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$ we'll be w/in .0004 of $\sqrt[3]{x}$.

e.g. $T_2(8.3) = 2.0246875$

where as $\sqrt[3]{8.3} = 2.024693...$