Power Series as Functions

Idea: can view a power series

$$\sum c_n (x-a)^n$$

as a function $f(x)$ on its interval of convergence.

amazingly: many well-known functions can be represented as power series

we already know one ex:

if $|x|<1$ then $f(x) = \frac{1}{1-x}$

is equal to $= \sum_{n=0}^{\infty} x^n = 1+x+x^2+\dots$

$$\boxed{= \sum_{n=1}^{\infty} 1 x^{n-1} = \frac{1}{1-x} \text{ by geo. series fmla}}$$

- of course if $|x| \geq 1$, $\sum x^n$ diverges and so is no longer $= \frac{1}{1-x}$.
- but we'll see: representing functions $f(x)$ as power series on even parts of their domain can be useful!

② ex: express $f(x) = \frac{1}{1+x^2}$ as a power series and find the interval of convergence.

Sol'n: we know $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$

as long as $|t|<1$.

So let $t = -x^2$ we get

$$\frac{1}{1-(-x^2)} = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

as long as $|-x^2|<1$, i.e. $x^2<1$, i.e. $|x|<1$

because $\sum_0^{\infty} (-x^2)^n$ is a geometric series, (with $r = -x^2$), this is as good as we can do: converges if and only if $|-x^2|<1$

i.e. interval of convergence is $(-1,1)$.

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ex: Same q for $f(x) = \frac{1}{x+2}$.

Sol'n: $\frac{1}{x+2} = \frac{1}{2+x} = \frac{1}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$

$= \frac{1}{2}\left(\frac{1}{1-(-\frac{x}{2})}\right)$

$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^n x^n$

$= \frac{1}{2}\left(1 - \frac{1}{2}x + \frac{1}{4}x^2 - \cdots\right)$

$= \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \cdots = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)^{n+1}x^n$

which converges iff $\left|-\frac{x}{2}\right| < 1$

i.e. $\left|\frac{x}{2}\right| < 1$

i.e. $-1 < \frac{x}{2} < 1$

i.e. $-2 < x < 2$

So interval of convergence in this case is $(-2,2)$.

4 ex: Same q for $f(x) = \frac{x^3}{x+2}$.

Sol'n: - we know we can distribute constants over series: $c\sum a_n = \sum c a_n$

- we also know for a fixed x in $(-2,2)$, the series $\sum_0^\infty (-1)^n (\frac{1}{2})^{n+1} x^n$ converges to $\frac{1}{x+2}$

- so... for x in $(-2,2)$ we must have:

$$x^3 \sum_0^\infty (-1)^n (\tfrac{1}{2})^{n+1} x^n = \sum_0^\infty x^3 (-1)^n (\tfrac{1}{2})^{n+1} x^n$$

$$= \sum (-1)^n (\tfrac{1}{2})^{n+1} x^{n+3}$$

converges to $= \frac{x^3}{x+2}$!

So: $\sum_0^\infty (-1)^n (\frac{1}{2})^{n+1} x^{n+3} = \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \cdots$

is a power series representation for $\frac{x^3}{x+2}$

on interval $(-2,2)$.

① - Recall: we think of a power series

$$\sum_{0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

as an "infinite polynomial".

- Polynomials are nice because they are ez to differentiate + integrate.

- turns out: same is true of power series!

$d/dx\,(x^2 + x) = 2x+1$
$\int x^2 + x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$
term by term...

Theorem If the power series $\sum_{0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, then the function f defined by:

$$f(x) = \sum_{0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is continuous and differentiable on $(a-R, a+R)$ and further:

(i) $f'(x) = \sum_{1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$

(ii) $\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$

$$= \left(c_0 (x-a) + \frac{c_1 (x-a)^2}{2} + \frac{c_2 (x-a)^3}{3} + \cdots \right) + C$$

② and the radius of convergence for both series is also R.

- won't prove theorem (beyond scope)
- Note: thm says: can differentiate + integrate power series term-by-term!
- Note: thm also says: radius R of convergence for differentiated and integrated series is same as original series, but what happens at the endpoints $x=a+R$ and $x=a-R$ may be different.
- can use thm to get power series rep'ns for new functions.

ex: Find a power series representation for $\frac{1}{(1-x)^2}$. What is the radius of convergence?

Sol'n: We know:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots \quad \text{if } \underbrace{|x|<1}_{R}$$

③

So:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}(1+x+x^2\cdots)$$

by theorem

i.e. $$\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots$$

$$= \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$$

thm also says: $R=1$ for this series too.
(can check: $\sum (n+1)x^n$ diverges if $x=1$ or -1,
so interval of convergence is $(-1,1)$).

(in this case: same as original interval, but in general can be diff).

ex: Find power series rep'n for $\frac{x}{(1-x)^2}$.

Soln: by above:

$$\frac{1}{(1-x)^2} = \sum_0^{\infty}(n+1)x^n = 1+2x+3x^2+\cdot \quad \text{if } |x|<1$$

so $$\frac{x}{(1-x)^2} = x\sum_0^{\infty}(n+1)x^n = x(1+2x+3x^2+\cdots)$$

$$= x+2x^2+3x^3+\cdots$$

$$= \sum_0^{\infty}(n+1)x^{n+1} \quad \text{for } |x|<1.$$

mult. by x
does not change radius

(4) ex: Same q for $\ln(1+x)$.

Soln: a new trick.

Observe: $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$

$$= \frac{1}{1-(-x)}$$

$$\overset{\text{geo series form}}{=} \sum_{n=0}^{\infty} (-x)^n$$

$$= \sum_{0}^{\infty} (-1)^n x^n$$

$$= 1 - x + x^2 - x^3 + \dots$$

for $|x|<1$

Hence $\ln(1+x) = \int \sum_{0}^{\infty} (-1)^n x^n$

$$= \int (1 - x + x^2 - x^3 + \dots)$$

$$= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots + C$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + C \quad \text{if } |x|<1$$

To solve for C at $x=0$

$\ln(1) = 0 + C$

i.e. $0 = C$. Hence: $\ln(1+x)$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}/n \; x^n \quad \text{if } |x|<1.$$

(won't check end points).

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ex: Same question for $f(x) = \tan^{-1}(x)$

Soln: we know:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{n=0}^{\infty} (-x^2)^n \qquad \text{if } |-x^2|<1 \text{ i.e. } |x|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

So: $\tan^{-1}x = \int 1 - x^2 + x^4 - x^6 + \cdots$ for $|x|<1$

$$= \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right) + C$$

To find C, set $x=0$:

$$\tan^{-1}0 = 0 + C$$

i.e. $0 = C$.

So $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \checkmark$$

Ex: evaluate $\int \frac{x}{1+x^3}\,dx$ as a power series. What is radius of convergence?

Sol'n: first we do $\frac{x}{1+x^3}$

We know:

$$\frac{x}{1+x^3} = x\left(\frac{1}{1+x^3}\right) = x\left(\frac{1}{1-(-x^3)}\right)$$

$$= x\left(\sum_0^\infty (-x^3)^n\right) \qquad \text{if } |x^3|<1 \text{ i.e. if } |x|<1$$

$$= x\left(\sum_0^\infty (-1)^n x^{3n}\right) = x(1-x^3+x^6-x^9+\cdots)$$

$$= \sum_0^\infty (-1)^n x^{3n+1}$$

$$= x - x^4 + x^7 - x^{10} + \cdots$$

for $|x|<1$.

Hence $\int \frac{x}{1+x^3}\,dx$

$$= \int x - x^4 + x^7 - x^{10} + \cdots \qquad \text{for } |x|<1$$

$$= \frac{1}{2}x^2 - \frac{1}{5}x^5 + \frac{1}{8}x^6 - \frac{1}{11}x^4 + \cdots + C$$

$$= \sum_0^\infty (-1)^n \frac{x^{3n+2}}{3n+2} + C \qquad \text{for } |x|<1$$

7 Note: it turns out

$$\int \frac{x}{1+x^3}\,dx = \frac{1}{6}\Big(-\ln(x^2-x+1) + 2\ln(x+1) + 2\sqrt{3}\,\tan^{-1}\Big(\frac{2x-1}{\sqrt{3}}\Big)\Big) + C$$

... but this is hard to solve for directly.

Our power series rep'n of this integral is good enough for many applications.

ex. Approximate $\int_0^{0.3} \frac{x}{1+x^3}\,dx$ using the previous power series, to within two decimal places.

Sol'n: from above:

$$\int \frac{x}{1+x^3} = \frac{1}{2}x^2 - \frac{1}{5}x^5 + \frac{1}{8}x^8 - \cdots + C$$

$$= \sum_0^\infty (-1)^n \frac{x^{3n+2}}{3n+2} + C \quad \text{for } |x| <$$

Hence $\int_0^{0.3} \frac{x}{1+x^3} = \left(\frac{1}{2}(.3)^2 - \frac{1}{5}(.3)^5 + \frac{1}{8}(.3)^8 - \cdots\right)$

$$- \cancel{\left(\frac{1}{2}0^2 - \frac{1}{5}0^5 + \frac{1}{8}0^8 - \cdots\right)}^{\,0}$$

$$= \frac{1}{2}(.3)^2 - \frac{1}{5}(.3)^5 + \frac{1}{8}(.3)^8 - \cdots$$

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- this is a convergent alternating series
- let's approximate it using first three terms:

$$\approx \overset{a_0}{\frac{1}{2}(.3)^2} - \overset{a_1}{\frac{1}{5}(.3)^5} + \overset{a_2}{\frac{1}{8}(.3)^8}$$

$$= .044522205$$

we know error is bounded by next term:

$$R_2 \leq b_{n+1} = \frac{1}{11}(.3)^{11} = 1.61 \times 10^{-7}$$

i.e. .0445... is w/in 1.61×10^{-7} of $\int_0^{0.3} \frac{x}{1+x^3}\,dx$ ... way better than two decimal places!