Class info:

21-122 Integration and Approximation

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Office hours: Mon 10:30 – noon Friday 2:30–4:00

Textbook: Calculus, Early Transcendentals
8th edition
Stewart

Assignments and Grades:

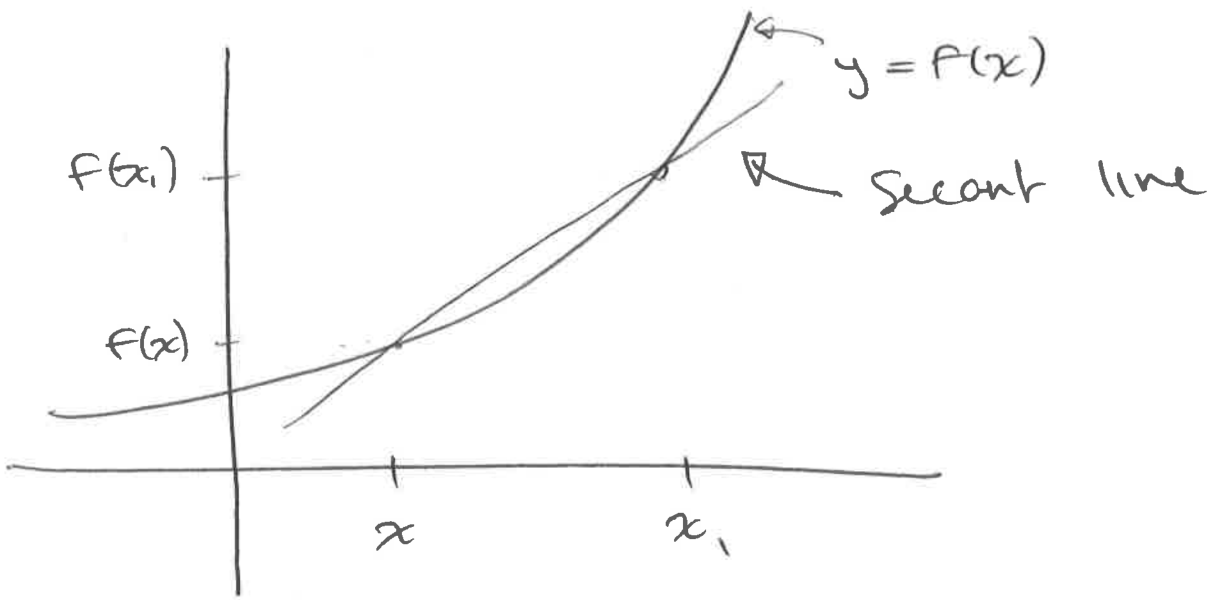
Weekly HW: 30%
Quizzes: 10%
Two midterms: 15% each
Final: 30%

- lowest HW and quiz score dropped
- no late HW
- staple your HW

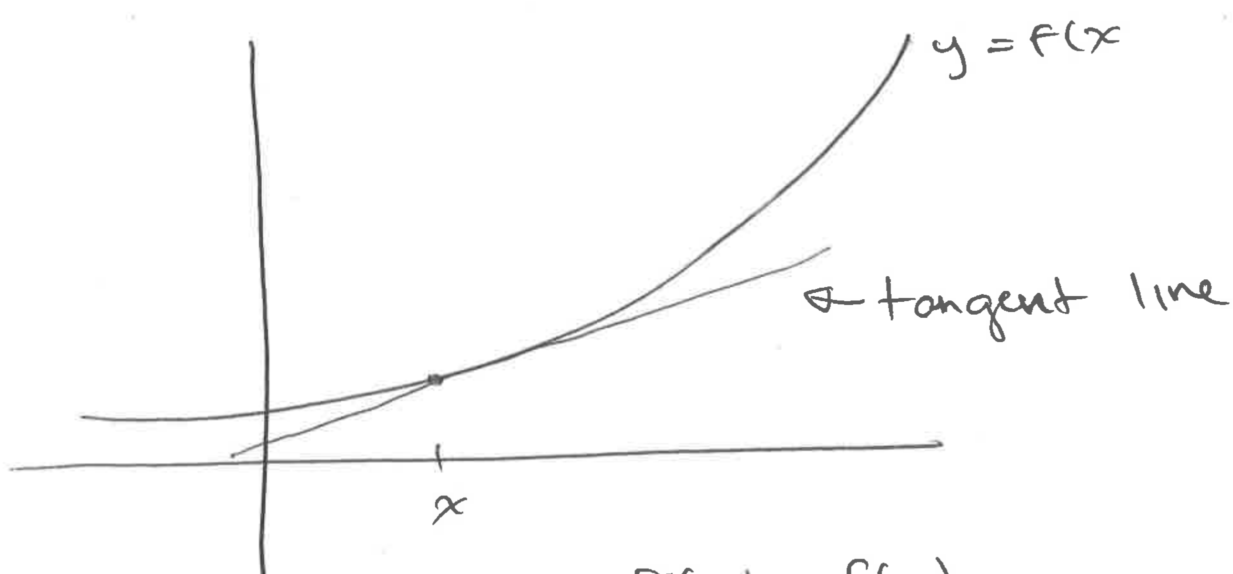
↳ will post HW, solutions, notes, grades, etc. on Canvas page.

Review of concepts:

$\boxed{\text{Derivatives}}$ Sps $f(x)$ is a (nice) function.



Slope is: $\dfrac{f(x_1) - f(x)}{x_1 - x}$



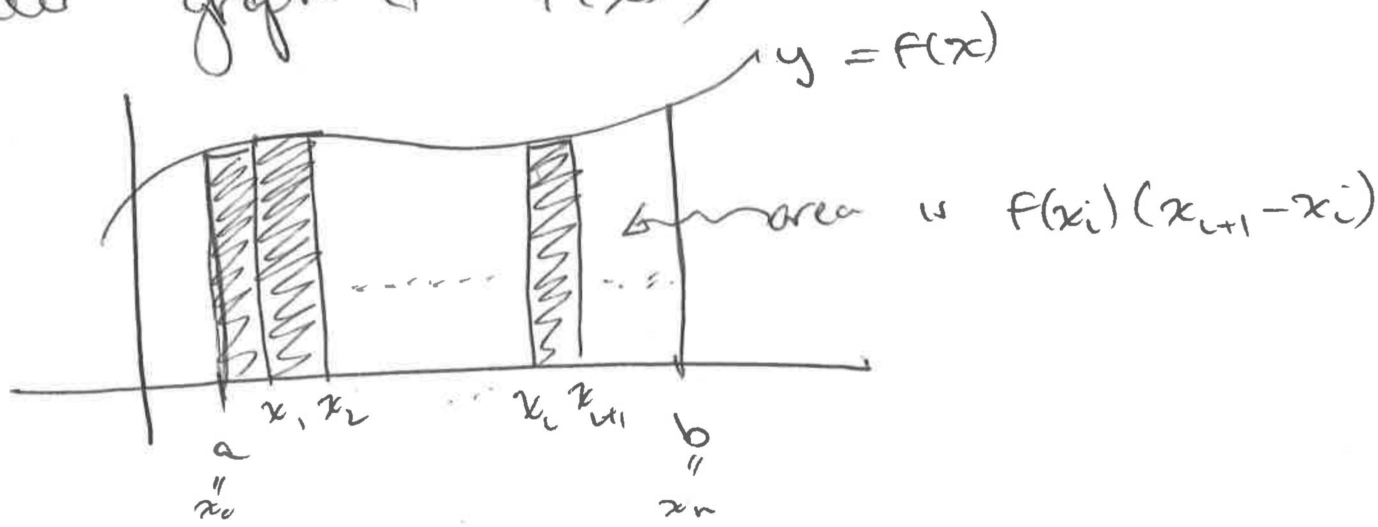
Slope is $\lim\limits_{x_1 \to x} \dfrac{f(x_1) - f(x)}{x_1 - x}$

we write $f'(x)$ or $\frac{df}{dx}$ for this limit (at every x where it exists)

$\boxed{\text{Antiderivatives}}$ Given functions $f(x)$, $F(x)$, if $F'(x) = f(x)$ we say $F(x)$ is an antiderivative of $f(x)$

we write $\int f(x)\,dx$ for general antiderivative of $f(x)$ aka indefinite integral of $f(x)$

$\boxed{\text{Integrals}}$ Given $f(x)$ and a partition of $[a,b]$ can compute a Riemann sum (approx. signed area under graph of $f(x)$)



Total area of rectangles

$$= \sum_{i=0}^{n-1} f(x_i)\,(x_{i+1} - x_i)$$

$\approx$ area under graph of $y = f(x)$

↳ can take lim of $x_{i+1} - x_i \to 0$ to get actual (signed) area below $f(x)$

aka the integral of $f(x)$ over $[a, b]$:

write: $$\int_a^b f(x)\,dx := \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

a priori: $\int f(x)\,dx$ and $\int_a^b f(x)\,dx$ are unrelated concepts.

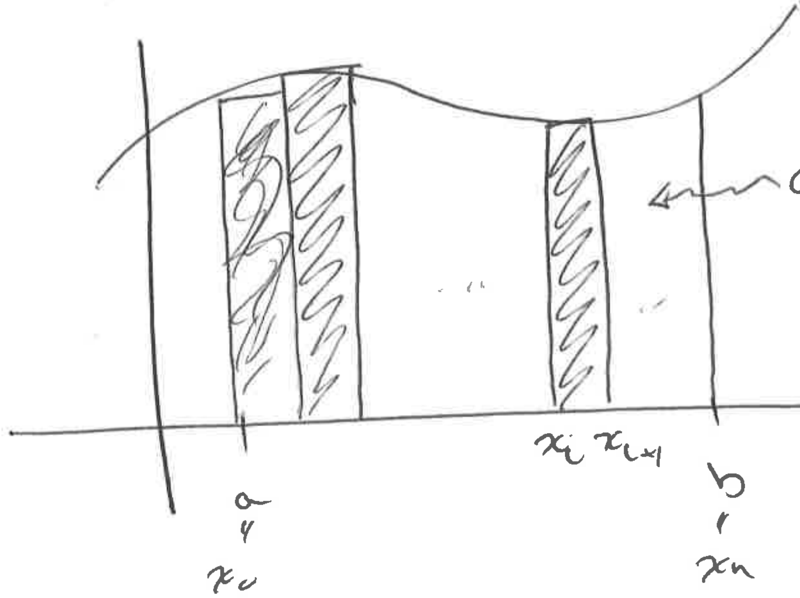
but FTC connects them!

Theorem (FTC): if $F'(x) = f(x)$

then $$\int_a^b f(x)\,dx = F(b) - F(a)$$

"Proof" of FTC:

$y = f(x) = F'(x)$



$$\begin{aligned} \text{area} &= f(x_i)(x_{i+1}-x_i) \\ &= F'(x_i)(x_{i+1}-x_i) \\ &\approx \frac{F(x_{i+1})-F(x_i)}{\cancel{x_{i+1}-x_i}}\cancel{(x_{i+1}-x_i)} \\ &= F(x_{i+1})-F(x_i) \end{aligned}$$

So sum of rectangles

$$\begin{aligned} &\approx \cancel{F(x_1)} - \underbrace{F(x_0)}_{F(a)} + \cancel{F(x_2)} - \cancel{F(x_1)} + \cancel{F(x_3)} - \cancel{F(x_2)} \\ &\quad + \cdots + \underbrace{F(x_n)}_{F(b)} - \cancel{F(x_{n-1})} \\ &\approx F(b) - F(a) \\ &\approx \text{area under } f(x) \text{ i.e } \int_a^b f(x)\,dx \end{aligned}$$

and $\approx$ becomes $=$ as $\Delta x \to 0$. ✓

FTC says: to find areas (i.e. integrals) need to know a lot of (anti) derivatives can compute a lot of derivs "by hand" from limit def'n

e.g. $$\frac{d}{dx} x^n = n x^{n-1} \qquad (n \neq 0)$$

$$\frac{d}{dx} \sin(x) = \cos(x)$$

$$\frac{d}{dx} e^x = e^x$$

$\vdots$ etc.

reversing any deriv. rule gives an integral rule:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \qquad (n \neq -1)$$

$$\int \cos x \, dx = \sin x + C$$

$$\int e^x \, dx = e^x + C$$

$\vdots$ etc.

---

ez to show $\frac{d}{dx}$ is a "linear operator"
i.e.

$$\frac{d}{dx} \left(a f(x) + b g(x) \right) = a f'(x) + b g'(x)$$

So $\int$ is linear too:

$$\int a f(x) + b g(x)\, dx$$
$$= a f(x) + b g(x)$$

---

also know the <u>chain rule</u>:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

which corresponds to integral rule:

$$\int f'(g(x))\, g'(x)\, dx$$
$$= f(g(x)) + C$$

sometimes called "u-sub" rule.

<u>ex</u>: $$\frac{d}{dx} \overset{f\,(g(x))}{\sin(x^3)} = \overset{f'\,(g(x))}{\cos(x^3)} \cdot \overset{g'(x)}{3x^2}$$
$$= 3x^2 \cos(x^3)$$

and in reverse:

$$\int 3x^2 \cos(x^3)\, dx$$

$$u = x^3$$
$$du = 3x^2\, dx$$

$$\int \cos(u)\, du = \sin(u) + C$$
$$= \sin(x^3) + C$$

also know the <u>product rule</u>:

$$\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$$

which gives rule:

$$\int f'(x)g(x) + f(x)g'(x)\,dx$$
$$= f(x)\,g(x) + C$$

can rearrange this as:

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

letting $u = f(x)$ so that $du = f'(x)\,dx$

$v = g(x)$ $dv = g'(x)\,dx$

can rewrite:

$$\int u\,dv = uv - \int v\,du$$

↳ called "integration by parts" rule.

Examples

① Find $\int x\sin(x)\,dx$

Sol'n: Let $f(x) = x \longrightarrow f'(x) = 1$

$g(x) = -\cos(x) \longleftarrow g'(x) = \sin(x)$

then $\int x\sin(x)\,dx = \int f(x)\cdot g'(x)$.

$\overset{IBP}{=} f(x)g(x) - \int f'(x)g(x)$

$= x\cdot(-\cos(x))$

$\quad - \int -\cos x\,dx$

$= -x\cos(x) + \sin(x) + C$

Check: $\frac{d}{dx}\left[-x\cos x + \sin x\right]$

$= -1\cdot\cos x + -x\cdot(-\sin x) + \cos x$

$= x\sin x$ ✓

↪ more std to use u's and v's.

$\int x\sin(x)\,dx$

$u = x \qquad du = 1\,dx$

$v = -\cos x \qquad dv = \sin x\,dx$

So $\int x \sin x \, dx = \int u \, dv$

$\overset{IBP}{=} uv - \int v \, du$

$= -x\cos x - \int -\cos x \, dx$

$\vdots$

$= -x\cos x + \sin x + C$ ✓

Could have tried:

$u = \sin x \qquad du = \cos x \, dx$

$v = \frac{1}{2}x^2 \qquad dv = x \, dx$

so then: $\int x \sin x \, dx = \int u \, dv$

$= uv - \int v \, du$

$= \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \, dx$

worse than before.

moral: judicious choice of u and dv important.

2 $\int \underline{x} \underline{\sin(x^2)} \underline{dx}$ $\quad \frac{1}{2} \int \underset{g'(x)}{2x} \underset{f(g(x))}{\sin(x^2)}\, dx$

Sol'n: u-sub!

$$u = x^2 \qquad du = 2x\,dx$$

$$\frac{1}{2} du = x\,dx$$

$$\int x \sin(x^2)\,dx = \int \frac{1}{2} \sin(u)\,du$$

$$= -\frac{1}{2} \cos(u)\,du$$

$$= -\frac{1}{2} \cos(x^2) + C$$

3 What about

$$\int x^2 \sin(x)\,dx \quad ?$$

Sol'n: back to IBP:

$$u = x^2 \qquad du = 2x\,dx$$

$$v = -\cos(x) \qquad dv = \sin(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$= -x^2 \cos(x) - \int 2x(-\cos x))\,dx$$

$$= -x^2 \cos x + \int 2x \cos(x)\,dx$$

$$= -x^2 \cos x + 2 \int x \cos x\,dx$$

now we do:

$$\int x \cos x \, dx$$

$u = x \qquad du = 1 dx$

$v = \sin x \qquad dv = \cos x \, dx$

$$= uv - \int v \, du$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x$$

So, going back:

$$\int x^2 \sin x \, dx$$

$$= -x^2 \cos x + 2(x \sin x + \cos x)$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x \checkmark$$

(4) Find $\int \ln(x) \, dx$

Sol'n: magic

$u = \ln x \qquad du = \frac{1}{x} dx$

$v = x \qquad dv = 1 \cdot dx$

$$\int \ln(x) \, dx = \int u \, dv$$

$= uv - \int v\,du$

$= x \ln x - \int x \frac{1}{x}\,dx$

$= x \ln x - \int 1\,dx$

$= x \ln x - x + C$

check: $\frac{d}{dx}(x\ln x - x)$

$= 1 \cdot \ln x + x \cdot \frac{1}{x} - 1$

$= \ln x + 1 - 1$

$= \ln x$ ✓

(5) Another variation:

$\int \sin^2 x\,dx$ i.e. $\int (\sin x)^2\,dx$

Sol'n: $u = \sin x$ $du = \cos x\,dx$

$v = -\cos x$ $dv = \sin x\,dx$

$\int \sin^2 x\,dx = \int u\,dv$

$= uv - \int v\,du$

$= -\sin x \cos x + \int \cos^2 x\,dx$

$= -\sin x \cos x + \int 1 - \sin^2 x\,dx$

$$= -\sin x\cos x + x - \int \sin^2 x\,dx$$

$$\Rightarrow 2\int \sin^2 x\,dx = -\sin x\cos x + x$$

$$\Rightarrow \int \sin^2 x\,dx = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C$$ ✓

Alternate sol'n: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

so: $\int \sin^2 x\,dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x$$

$\sin 2x \to = 2\sin x\cos x$

$$= \frac{1}{2}x - \frac{1}{2}\sin x\cos x$$ ✓

⑥ Definite integrals are just as ever:

e.g.

$$\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x\Big|_0^{\pi/2}$$

$$= \pi/4 - \frac{1}{2}\cdot 0$$

$$-(0-0)$$

$$= \pi/4$$

7.2 (More) Trig Integrals

ex (1) What about

$$\int \sin^3 x \, dx ?$$

IBP looks bad.

Instead:

save a factor of $\sin x$

$$= \int \sin^2 x \sin x \, dx$$

rewrite in terms of $\cos x$

$$= \int (1-\cos^2 x) \sin x \, dx$$

$u = \cos x \qquad du = -\sin x \, dx$

$\qquad\qquad\qquad -du = \sin x \, dx$

$$= \int -(1-u^2) \, du$$

$$= \int u^2 - 1 \, du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

↳ this kind of trick often works when integrand is of form $\sin^m x \cos^n x$

Guide: to find $\int \sin^m x \cos^n x$

① if m odd, save a factor of $\sin x$ and put $\sin^{m-1} x$ in terms of $\cos x$ using $\sin^2 x = 1 - \cos^2 x$
Then let $u = \cos x$

② if n odd, save a factor of $\cos x$ and let $u = \sin x$

③ if both m, n even use power reducing identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

ex ② $\int \sin x \cos^5 x \, dx$

$= \int \sin x \cdot \cos^4 x \cdot \cos x \, dx$

$= \int \sin x \cdot (1 - \sin^2 x)(1 - \sin^2 x) \cdot \cos x \, dx$

$u = \sin x \qquad du = \cos x$

$= \int u (1 - u^2)(1 - u^2) \, du$

$$= \int u(1-2u^2+u^4)\,du$$
$$= \int u - 2u^3 + u^5 \;\; du$$
$$= \frac{1}{2}u^2 - \frac{1}{2}u^4 + \frac{1}{6}u^6 + C$$
$$= \frac{1}{2}\sin^2 x - \frac{1}{2}\sin^4 x + \frac{1}{6}\sin^6 x + C.$$

3 $\int \sin^2 x \cos^2 x \, dx$

$$= \int \frac{1}{2}(1-\cos 2x)\frac{1}{2}(1+\cos 2x)\,dx$$
$$= \int \frac{1}{4}(1-\cos^2 2x)\,dx$$
$$= \frac{1}{4}\int \sin^2 2x \, dx$$
$$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\,dx$$
$$= \frac{1}{8}\int 1-\cos 4x \, dx$$
$$\vdots$$
$$= \frac{1}{8}x - \frac{1}{32}\sin 4x + C \checkmark$$

Similar tricks work for integrals of the form

$$\int \tan^m x \sec^n x \, dx$$

Remember:
$\frac{d}{dx} \tan x = \sec^2 x$
$\frac{d}{dx} \sec x = \sec x \tan x$
$\tan^2 x = \sec^2 x - 1$
$\sec^2 x = \tan^2 + 1$

1. if n even, save a factor of $\sec^2 x$ and let $u = \tan x$
2. if m odd, save a factor of $\sec x \tan x$, and let $u = \sec x$
3. if neither... be clever!

<u>ex</u>: 3. $\int \tan x \sec^3 x \, dx$

$= \int (\sec x \tan x) \sec^2 x \, dx$

$u = \sec x$
$du = \sec x \tan x \, dx$

$= \int u^2 \, du$

$= \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C$ ✓

4 a spicy one:

$$\int x \tan^2 x \, dx$$

Sol'n: looks like IBP:

$u = x \qquad du = dx$

$dv = \tan^2 x \, dx \qquad v = \int \tan^2 x \, dx$

$= \int \sec^2 x - 1 \, dx$

$= \tan x - x$

so $\int x \tan^2 x \, dx = \int u \, dv$

$= uv - \int v \, du$

$= x \tan x - x^2 - \int \tan x - x \, dx$

$= x \tan x - x^2 - \ln|\sec x| + \frac{1}{2}x^2 + C$

$= x \tan x - \frac{1}{2}x^2 - \ln|\sec x| + C$ ✓

Recall:

$\int \tan x$

$= \ln|\sec x| + C$

7.3 Trig Subs

↳ can also exploit trig identities in integrals that don't explicitly involve trig functions
↳ magic of ... trig subs!

A trig sub is a kind of "inverse u-sub"

ex: Consider the integrals:

① $\int_0^1 2t(1+t^2)\,dt$ ② $\int_1^2 u\,du$

actually evaluate the same.
translate ① → ② by regular u-sub:

$u = 1+t^2$
$du = 2t\,dt$

as t varies 0 to 1
u varies 1 to 2

so int becomes

$$\int_{u=1}^{2} u\,du$$

but can also translate ② → ①
by "inverse" u-sub

$u = 1 + t^2$ as u varies 1 to 2
$du = 2t\,dt$ t varies 0 to 1

so int becomes

$$\int_{t=0}^{1} 2t(1+t^2)\,dt$$

(can check: $\int_0^1 2t(1+t^2)dt = \int_1^2 u\,du$
$= 3/2$)

↳ sometimes (though not here) expanding a variable (u) to a function ($1+t^2$) like this actually simplifies an integral by allowing us to exploit trig identities.

↳ Trig subs are examples of this kind of inverse substitution

↳ useful for evaluating integrals with following expressions:

if you see:	make sub:

$\sqrt{a^2-x^2}$ $\quad x = a\sin\theta$ (restrict: $-\pi/2 \le \theta \le \pi/2$)

$\sqrt{a^2+x^2}$ $\quad x = a\tan\theta$ ($-\pi/2 < \theta < \pi/2$)

$\sqrt{x^2-a^2}$ $\quad x = a\sec\theta$ ($0 \le \theta < \pi/2$)

so inverse functions
$\theta = \sin^{-1}\left(\frac{x}{a}\right)$
$\theta = \tan^{-1}\left(\frac{x}{a}\right)$
$\theta = \sec^{-1}\left(\frac{x}{a}\right)$
exist

subs allow us to use trig idents

$$\sin^2\theta + \cos^2\theta = 1$$

and $\tan^2\theta + 1 = \sec^2\theta$

to simplify expressions

ex's ① Find $\int \sqrt{1-x^2}\,dx$

Sol'n. Let $x = \sin\theta$ $\quad (\theta \in [-\pi/2, \pi/2])$

$dx = \cos\theta\, d\theta$

so int becomes

$$\int \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$$

$$\int \sqrt{\cos^2\theta}\,\cos\theta\, d\theta$$

always ≥ 0 by restriction on θ

$$= \int |\cos\theta| \cos\theta \, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

$$= \int \tfrac{1}{2}(1 + \cos 2\theta) \, d\theta$$

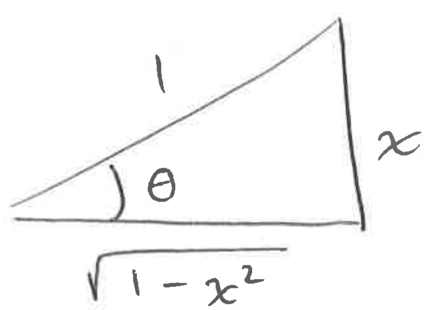
$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{\cancel{4}\,2}\sin\theta\cos\theta + C$$

not done! need to get in terms of x.

ez way: draw a triangle diagram

$x = \sin\theta$ so:



hence $\cos\theta = \sqrt{1-x^2}$

and $\theta = \sin^{-1}(x)$

$$= \tfrac{1}{2}\sin^{-1}(x) + \tfrac{1}{2}x\sqrt{1-x^2} + C$$

2 $\int x\sqrt{1-x^2}\,dx$

... regular u-sub

$u = 1-x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$= \int -\frac{1}{2}\sqrt{u}\,du$

$= -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$ ✓

3 $\int \frac{1}{(x^2+4)^2}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$= \int \frac{2\sec^2\theta}{(4\tan^2\theta+4)^2}\,d\theta$

$= \int \frac{2\sec^2\theta}{[4(\tan^2\theta+1)]^2}\,d\theta$

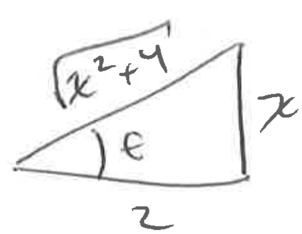
$= \int \frac{2\sec^2\theta}{16(\sec^2\theta)^2}$

$= \frac{1}{8}\int \cos^2\theta\,d\theta$

$= \frac{1}{8}\left(\frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta\right) + C$

but now:

$x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2}$



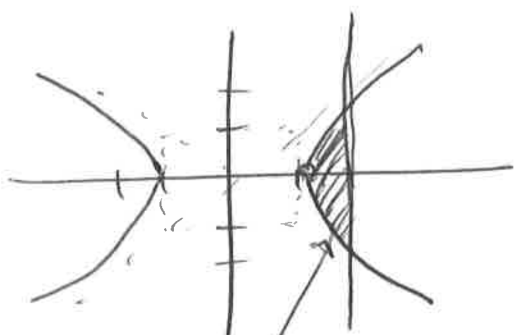
$\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$ $\qquad \cos\theta = \dfrac{2}{\sqrt{x^2+4}}$

so $= \dfrac{1}{16}\tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{1}{16}\cdot\dfrac{2x}{x^2+4} + c$ ✓

a spicy one:

(4) Find area bounded by curves

$x^2 - y^2 = 1$ and $x = 2$



area we want

$= 2 \times$ (top half)

$= 2 \times$ (area under $y = \sqrt{x^2-1}$)

$= 2\int_1^2 \sqrt{x^2-1}\, dx$

$x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$\int \sqrt{x^2-1}\, dx \longrightarrow \int \sqrt{\sec^2\theta - 1}\, \sec\theta\tan\theta\, d\theta$

$= \int \sqrt{\tan^2\theta}\ \sec\theta \tan\theta \, d\theta$

$= \int \sec\theta \tan^2\theta \, d\theta$

$= \int \sec\theta (\sec^2\theta - 1) \, d\theta$

$= \int \sec^3\theta - \int \sec\theta$

$\int \sec^3\theta$ $\downarrow$ by parts

$\int \sec\theta \rightarrow \ln|\sec\theta + \tan\theta|$

$u = \sec\theta \qquad dv = \sec^2\theta$

$du = \sec\theta \tan\theta \qquad v = \tan\theta$

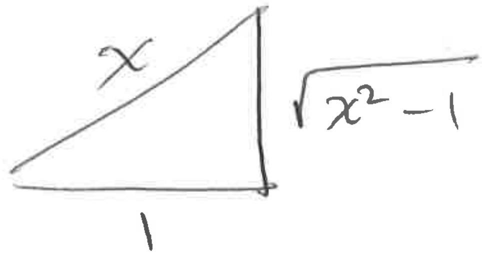
$\int \sec^3\theta = \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta$

So overall:

$\int \sec\theta \tan^2\theta \, d\theta = \sec\theta \tan\theta - \int \sec\theta \tan^2\theta \, d\theta - \ln|\sec\theta + \tan\theta|$

$\Rightarrow \int \sec\theta \tan^2\theta \, d\theta = \frac{1}{2} \sec\theta \tan\theta - \frac{1}{2} \ln|\sec\theta + \tan\theta| + C$

now: $x = \sec\theta$



so $\tan\theta = \sqrt{x^2-1}$

so $$\int_1^2 \sqrt{x^2-1}\, dx = \frac{1}{2}\left(x\sqrt{x^2-1} - \ln\left|\sqrt{x^2-1}+x\right|\right)\Bigg|_1^2$$

$$= \frac{1}{2}\left(2\sqrt{3} - \ln\left(\sqrt{3}+2\right)\right)$$

$$-\frac{1}{2}\left(1\cdot\cancel{0 - \ln 1}\right)$$

$$= \sqrt{3} - \frac{1}{2}\ln\left(\sqrt{3}+2\right) \checkmark$$